Transformational Geometry

…transforming instruction?

Henri Picciotto
www.MathEducation.page
henri@MathEducation.page
blog.MathEducation.page
Twitter: @hpicciotto

Common Core (2010)

A key innovation of the CCSSM in geometry is the redefinition of congruence on a transformational basis.

(and likewise for similarity)

Good idea!

- ◊ Connections to algebra:
  - Functions: composition, inverses, notation, …
  - Transformations of graphs
- ◊ Geometric interpretation of complex numbers
- ◊ Introduction to matrices
- ◊ Similarity of curves
- ◊ Symmetry

Pedagogically

a more intuitive foundation

Minimal interpretation of the CCSSM

1. Use transformations to justify SSS, SAS, ASA*
2. Continue along the traditional path

* but how?

Ambitious interpretation

Rethinking the course, with transformations throughout

Obstacles

“It’s not rigorous”

Often introduced strictly as special cases on the Cartesian plane.

What should a teacher know?

◊ Minimal
  - definitions and assumptions
  - basic theorems about transformations
  - triangle congruence and similarity

◊ Ambitious
  - symmetry definitions of special figures
  - transformational approach to proof throughout
  - related topics in subsequent courses

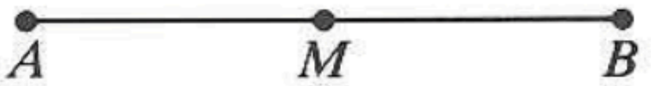
From a typical geometry textbook:

Theorem 2-1 *Midpoint Theorem*

If M is the midpoint of $\overline{AB}$, then $AM = \frac{1}{2}AB$ and $MB = \frac{1}{2}AB$.

Given: M is the midpoint of $\overline{AB}$.

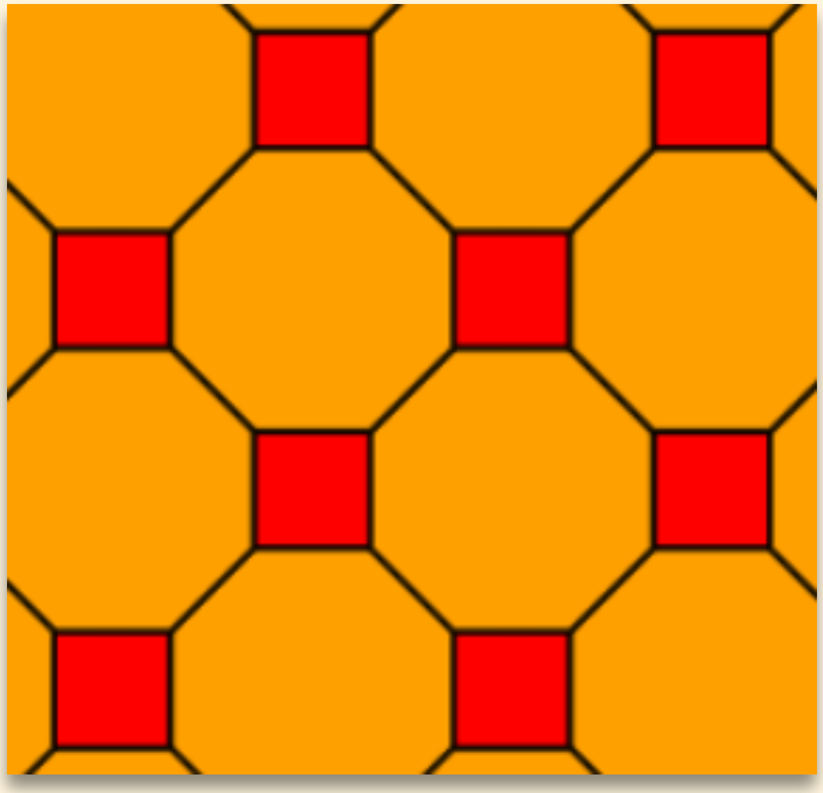
Prove: $AM = \frac{1}{2}AB$; $MB = \frac{1}{2}AB$

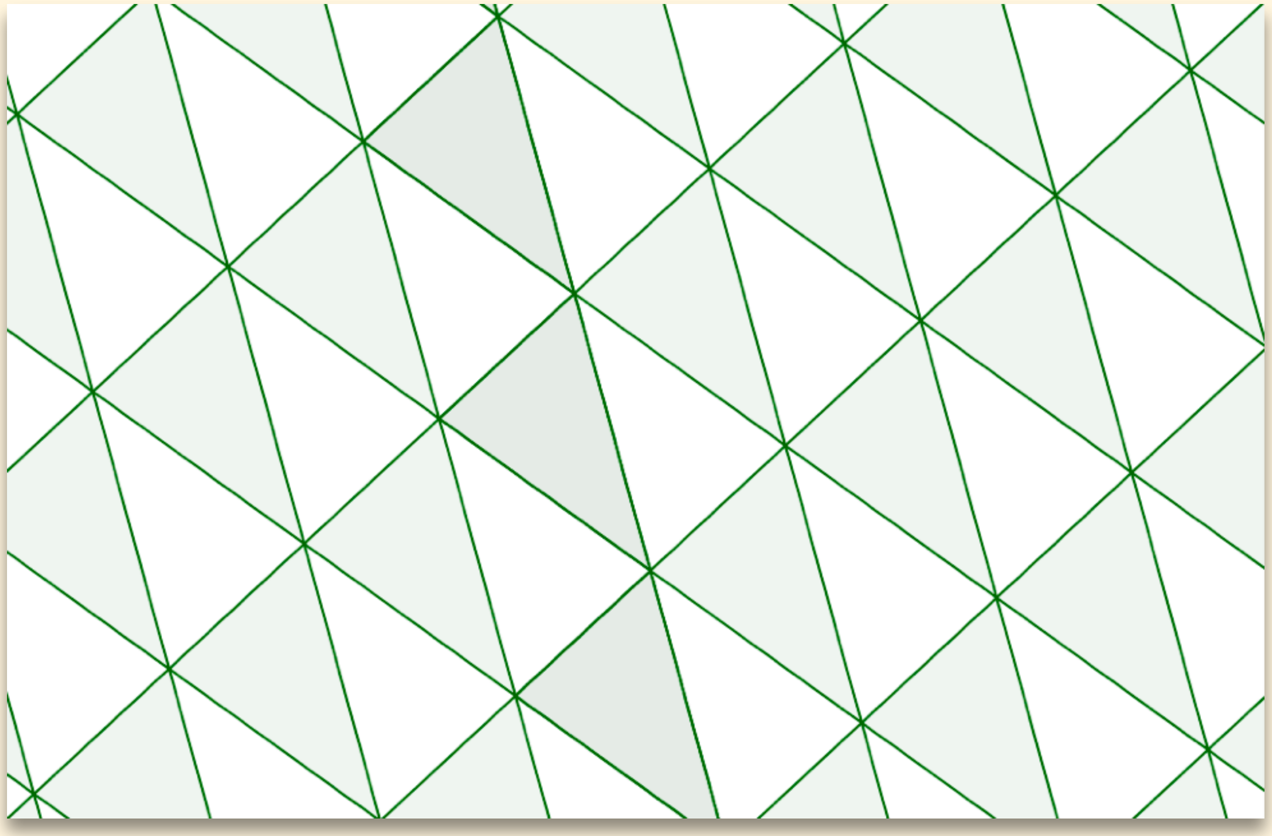


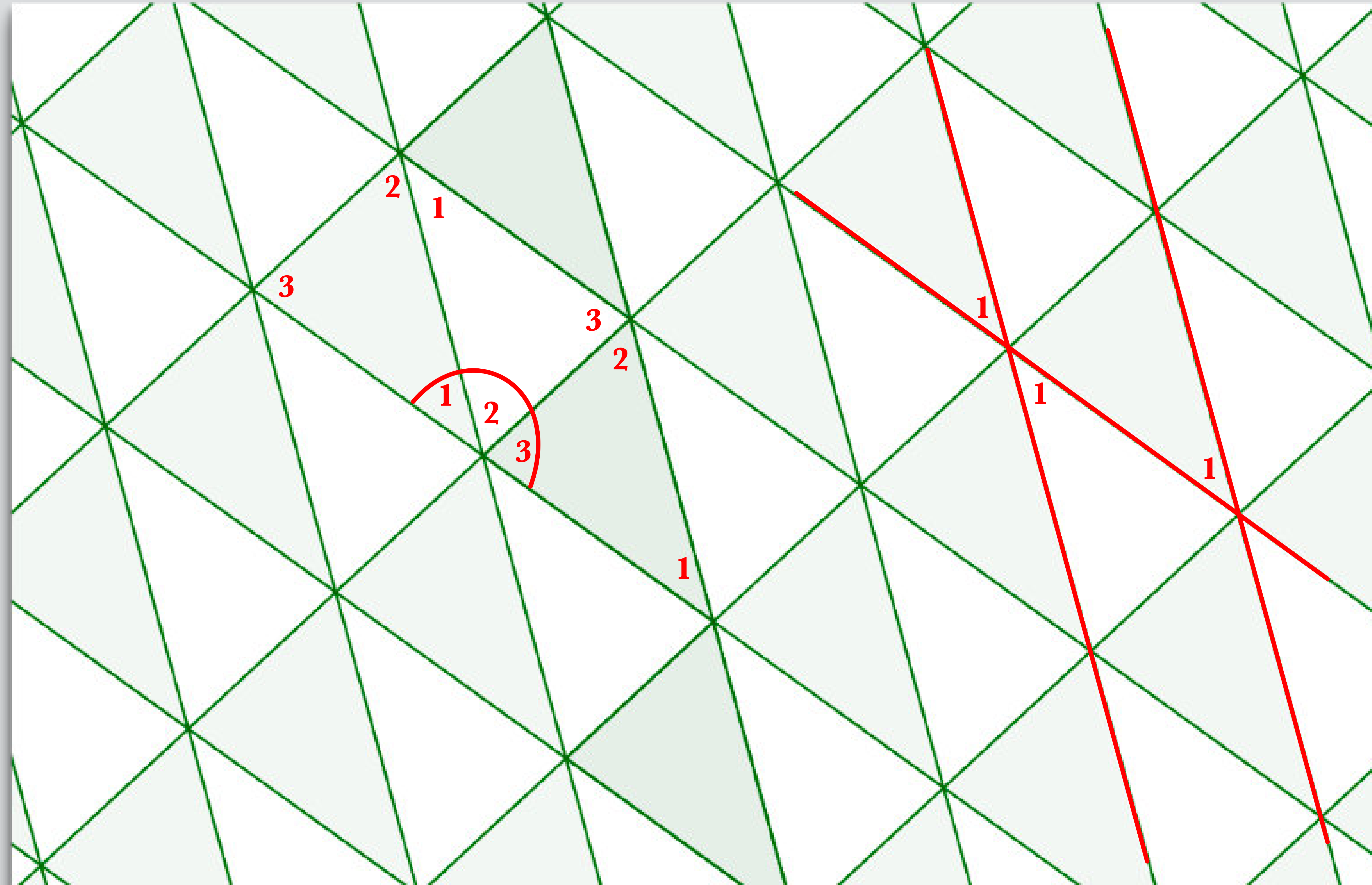
Proof:

Statements	Reasons
1. M is the midpoint of $\overline{AB}$.	1. Given
2. $\overline{AM} \cong \overline{MB}$, or $AM = MB$	2. Definition of midpoint
3. $AM + MB = AB$	3. Segment Addition Postulate
4. $AM + AM = AB$, or $2AM = AB$	4. Substitution Prop. (Steps 2 and 3)
5. $AM = \frac{1}{2}AB$	5. Division Prop. of =
6. $MB = \frac{1}{2}AB$	6. Substitution Prop. (Steps 2 and 5)

Tiling the Plane





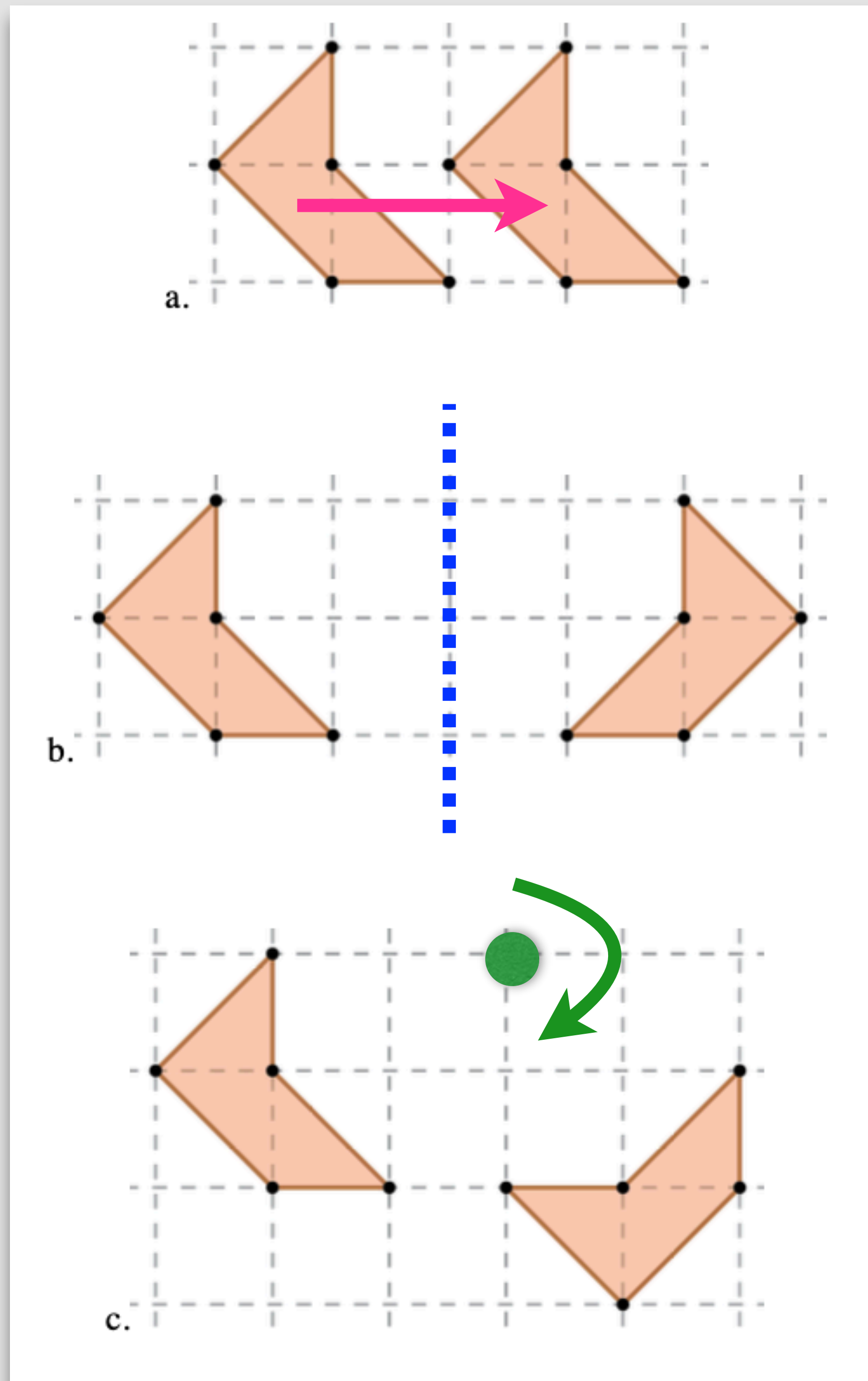


2
1
3
3
2
1
2
3
1
1
1
1

The Basics

Rigid motions (*isometries)* are transformations that preserve distance and angle measure.

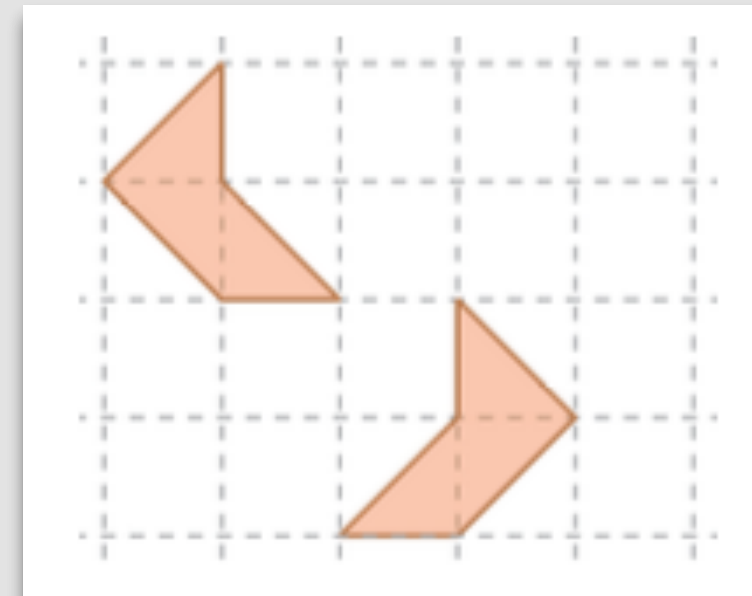
Two figures are *congruent* if one is the image of the other in a sequence of isometries.

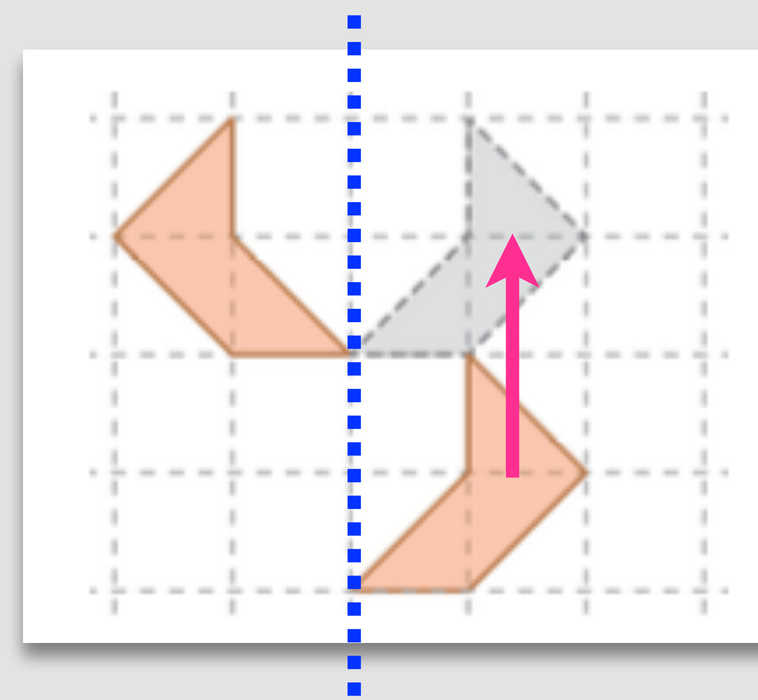


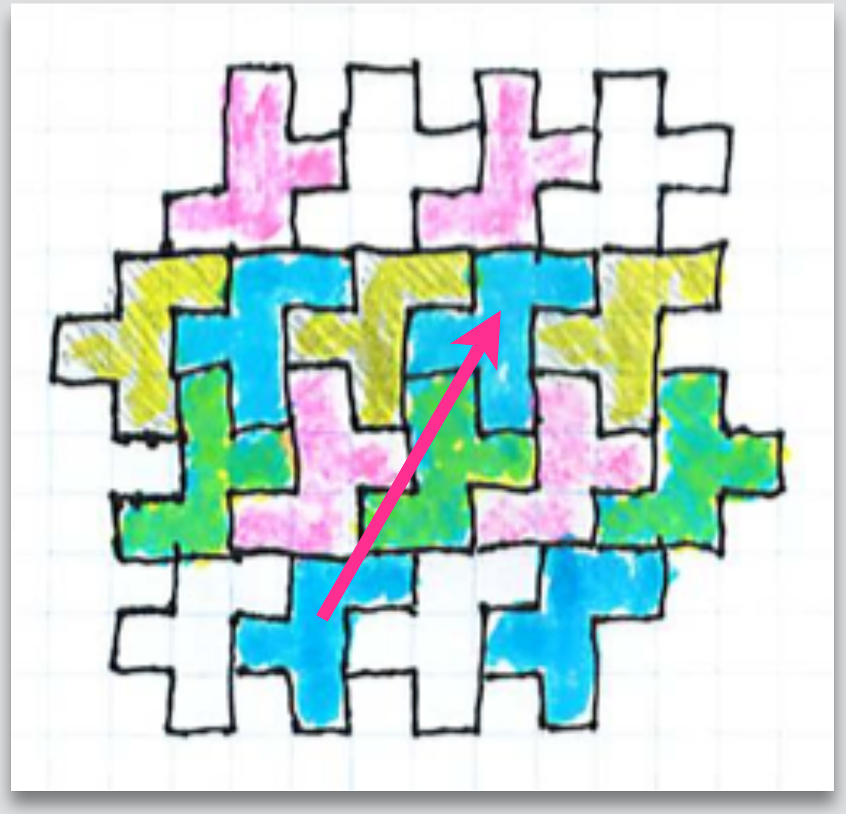
Translation

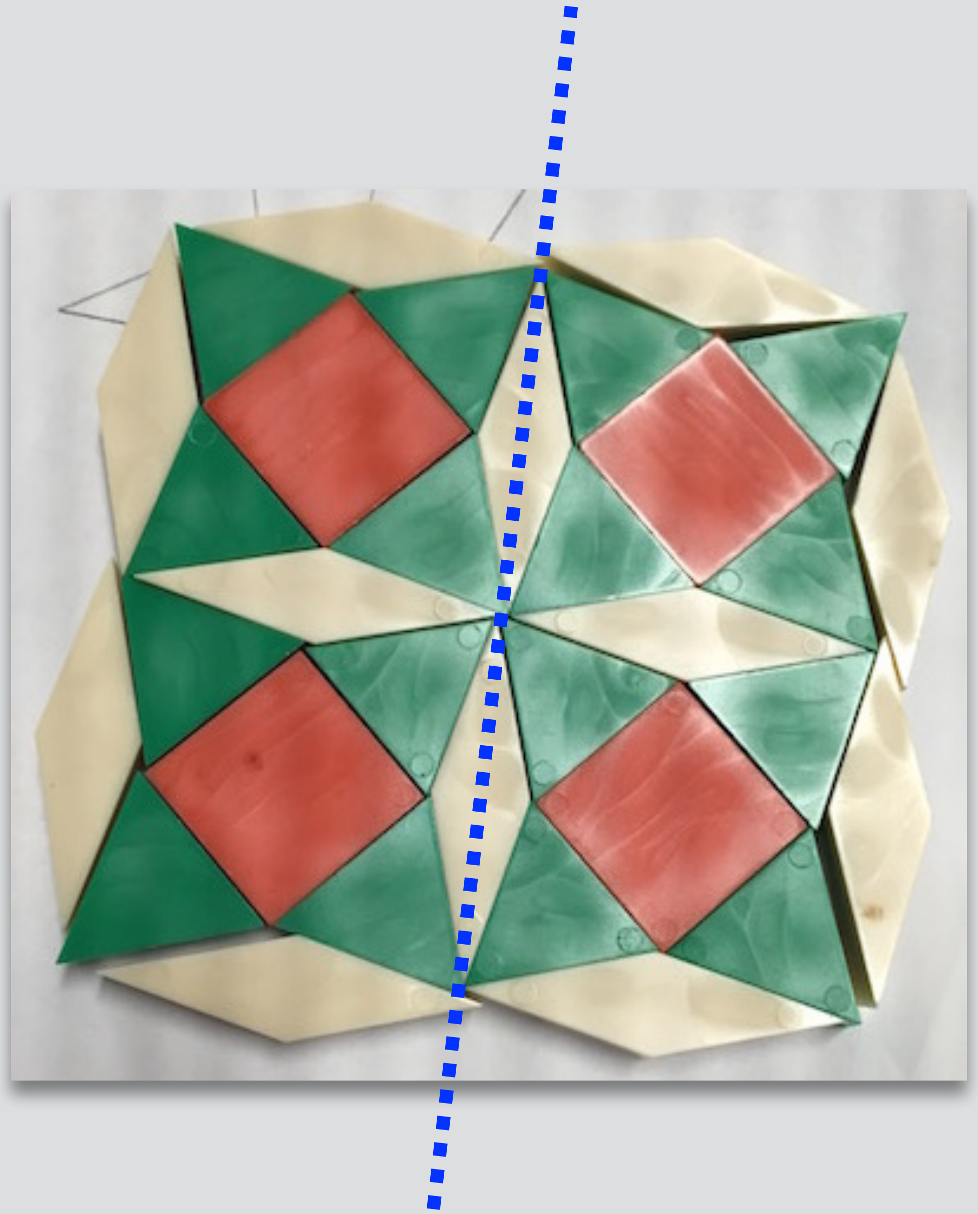
Reflection

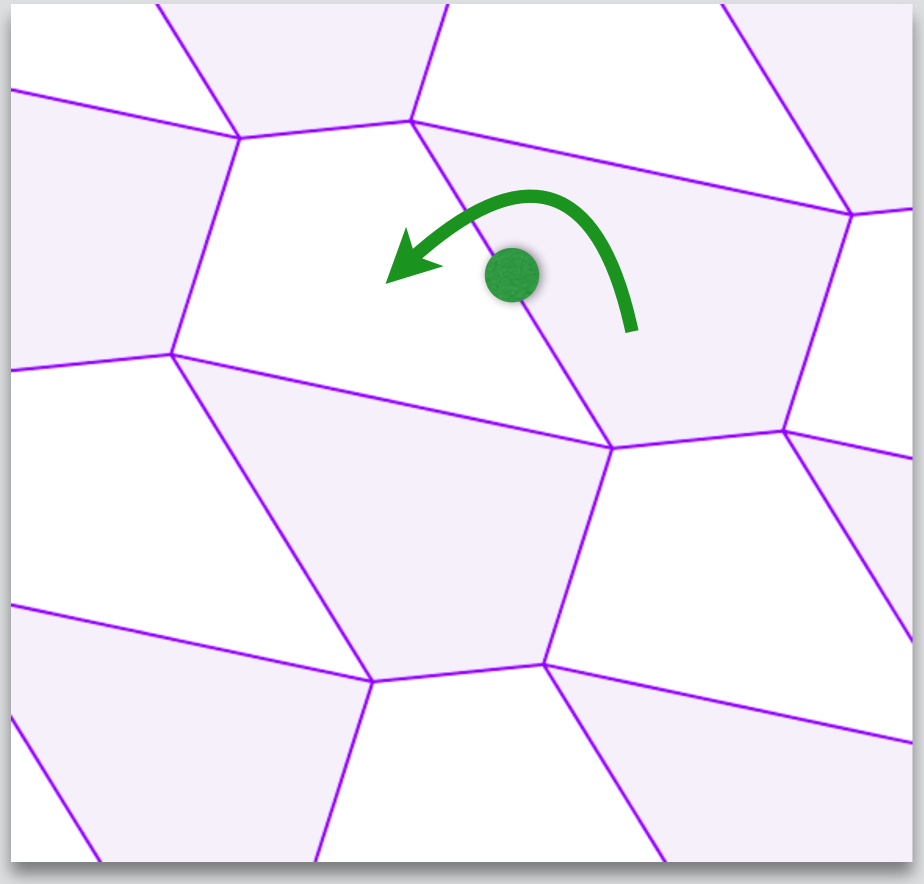
Rotation







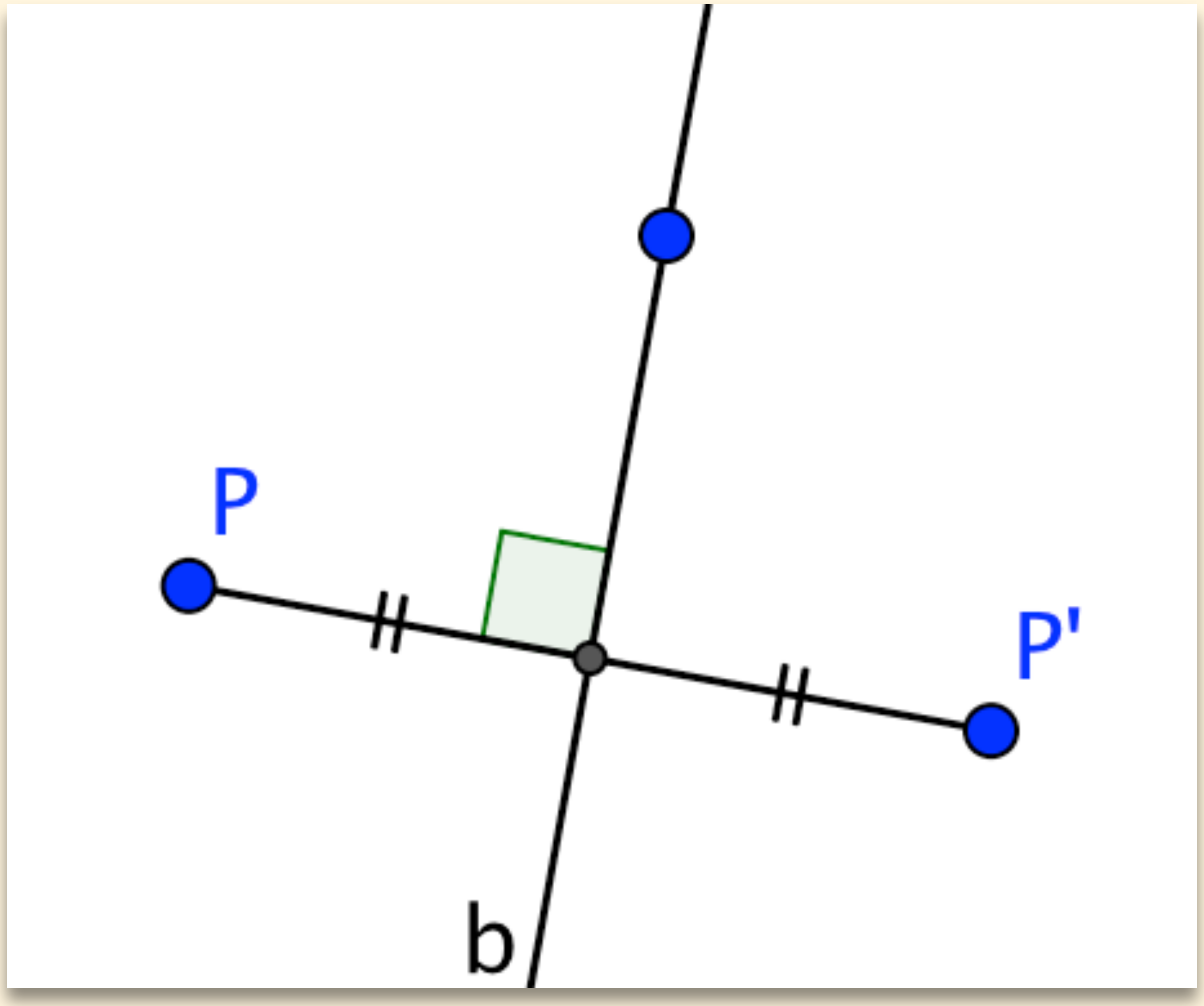




Precise definitions of reflection, rotation, and translation

Reflection

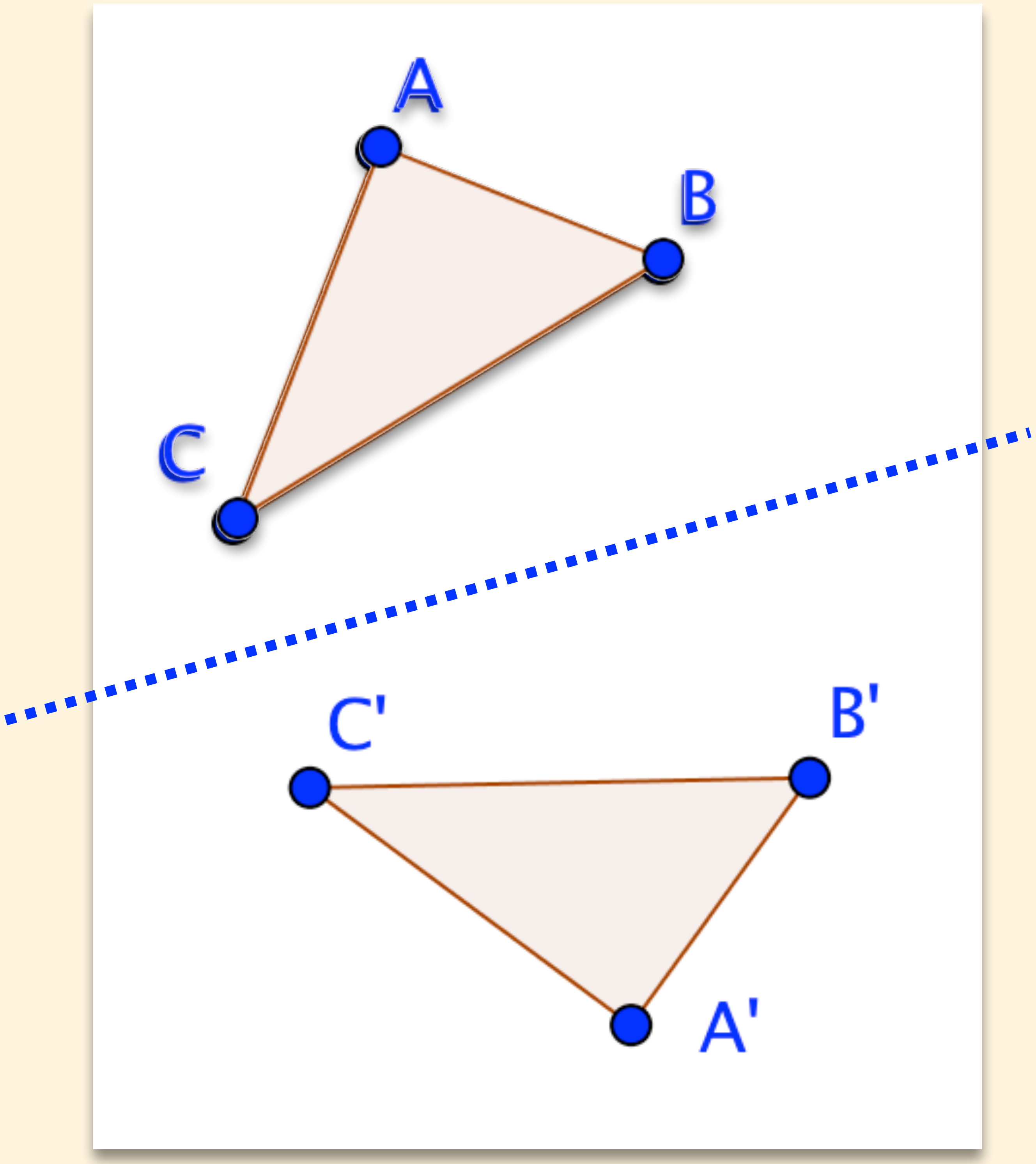
A reflection in a line *b* maps any point on *b* to itself, and any other point P to a point P' so that *b* is the perpendicular bisector of PP'.



What is preserved

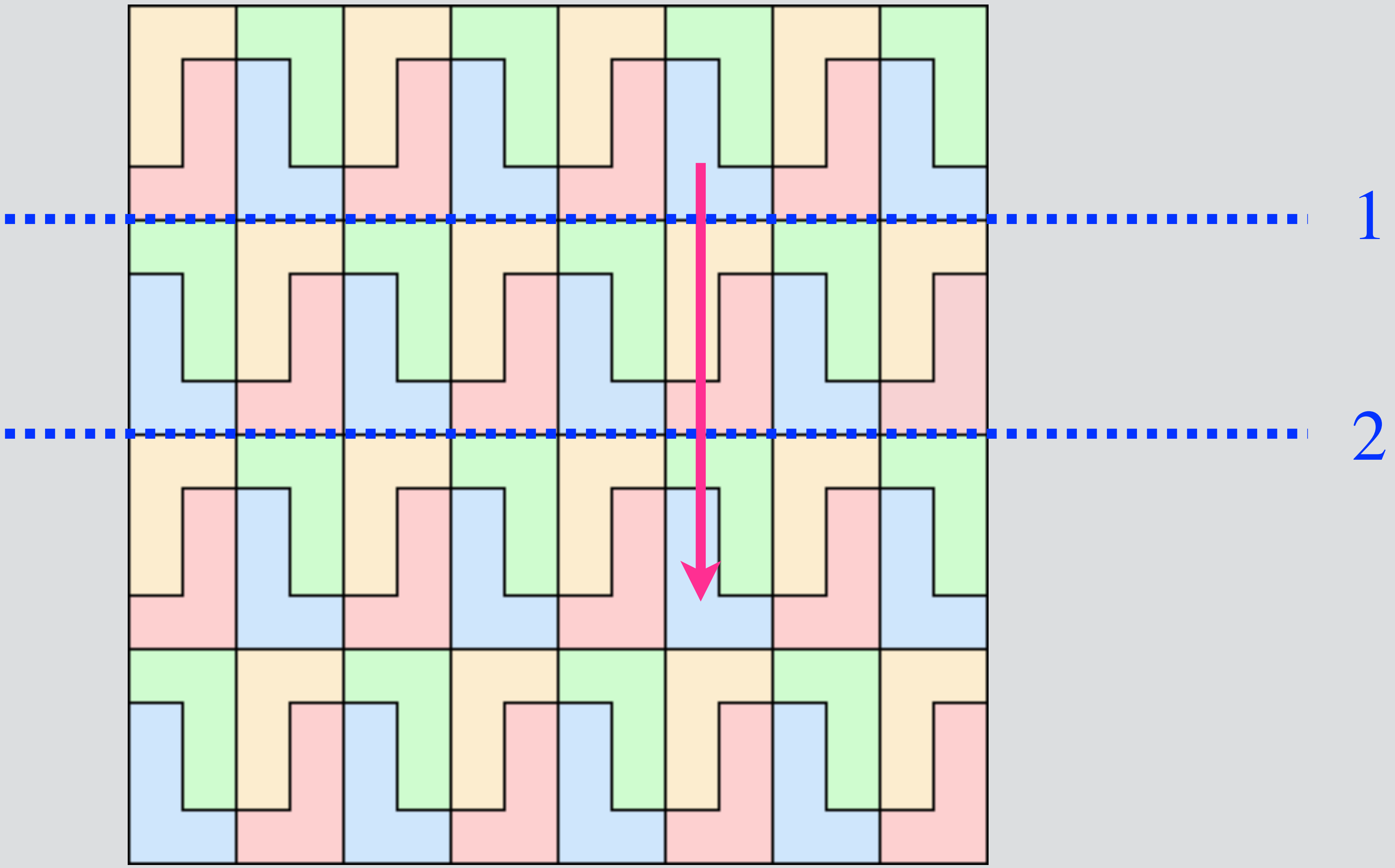
- Distance
- Angle measure
- Orientation?

Orientation

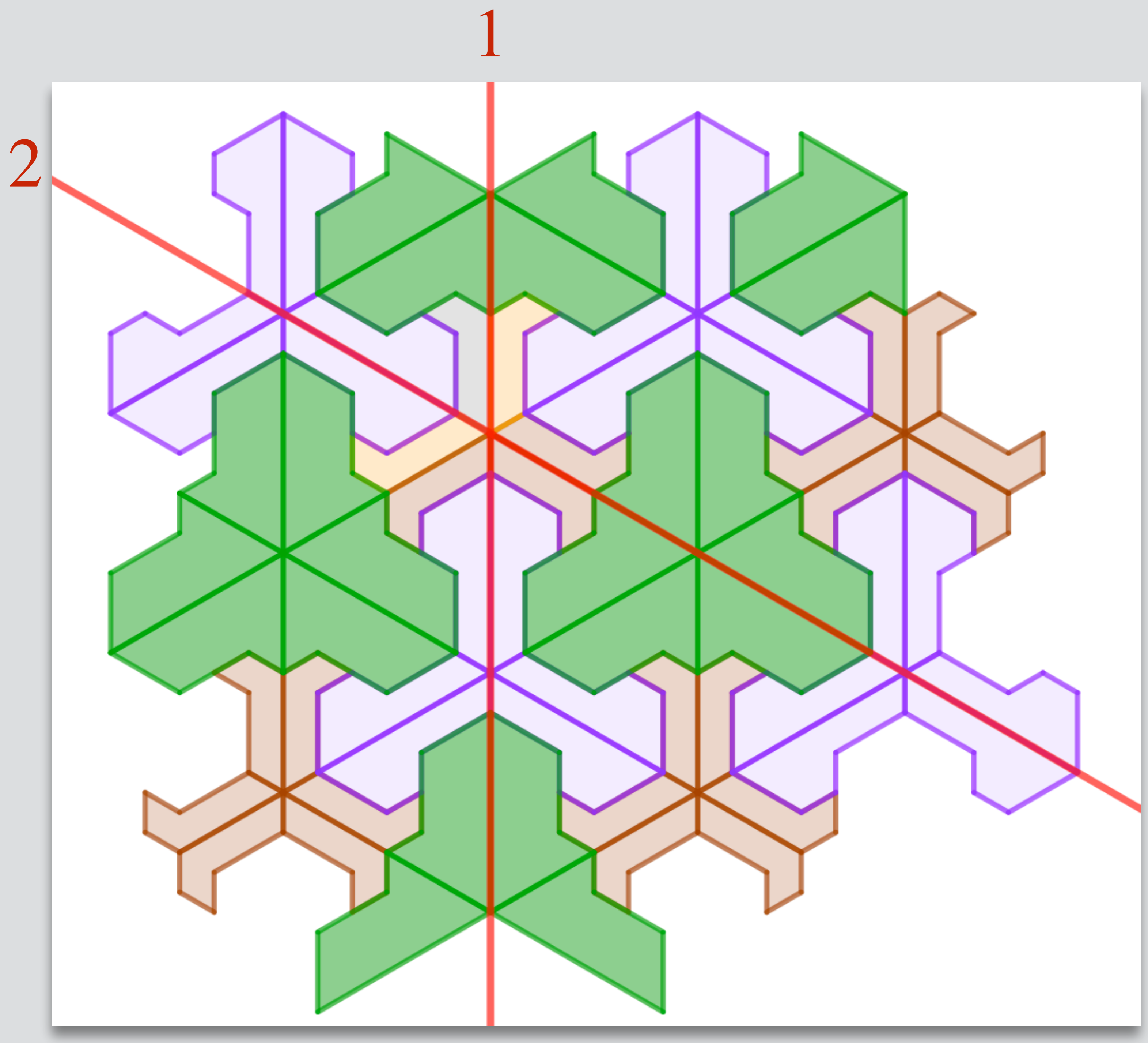


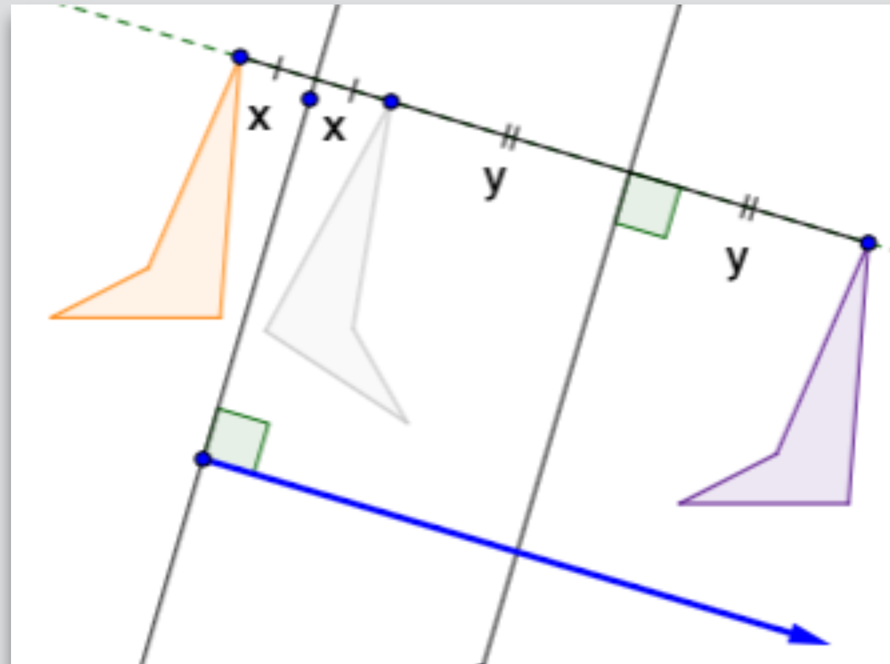
Composition
of two reflections

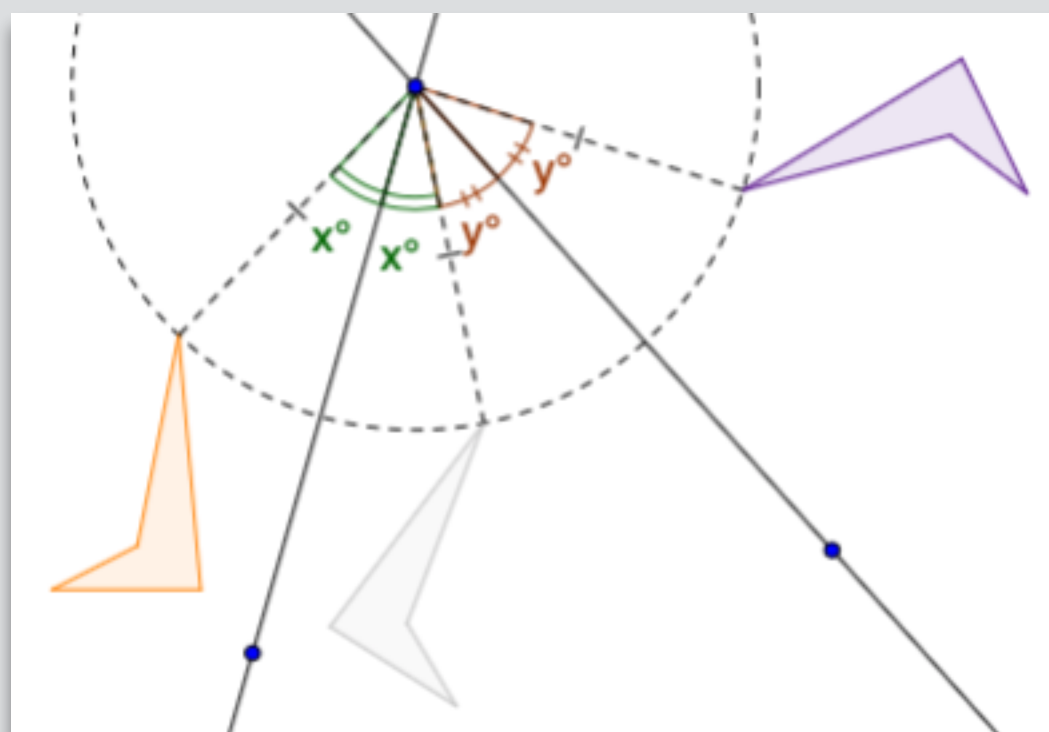
The composition of two reflections in parallel lines is a translation.



The composition of two reflections in intersecting lines is a rotation around their intersection.




x
x
y
y


x°
x°
y°
y°

(We can think of any translation or rotation as the composition of two reflections.)

Postulate:

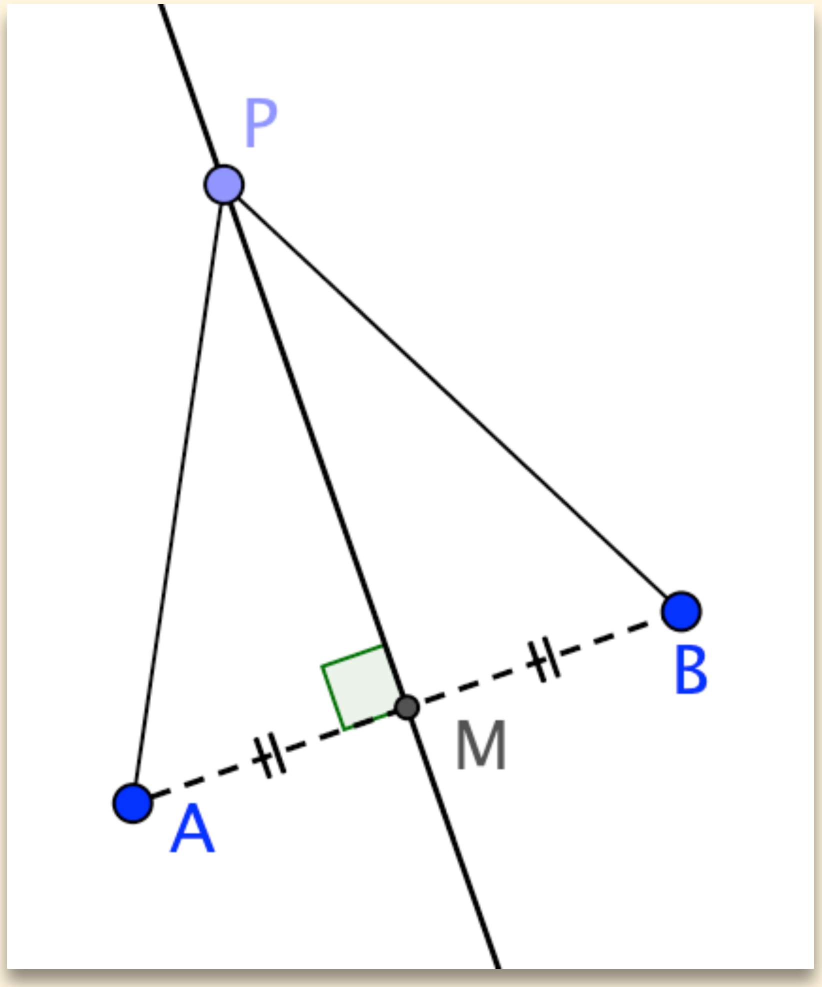
Reflection preserves distance and angle measure.

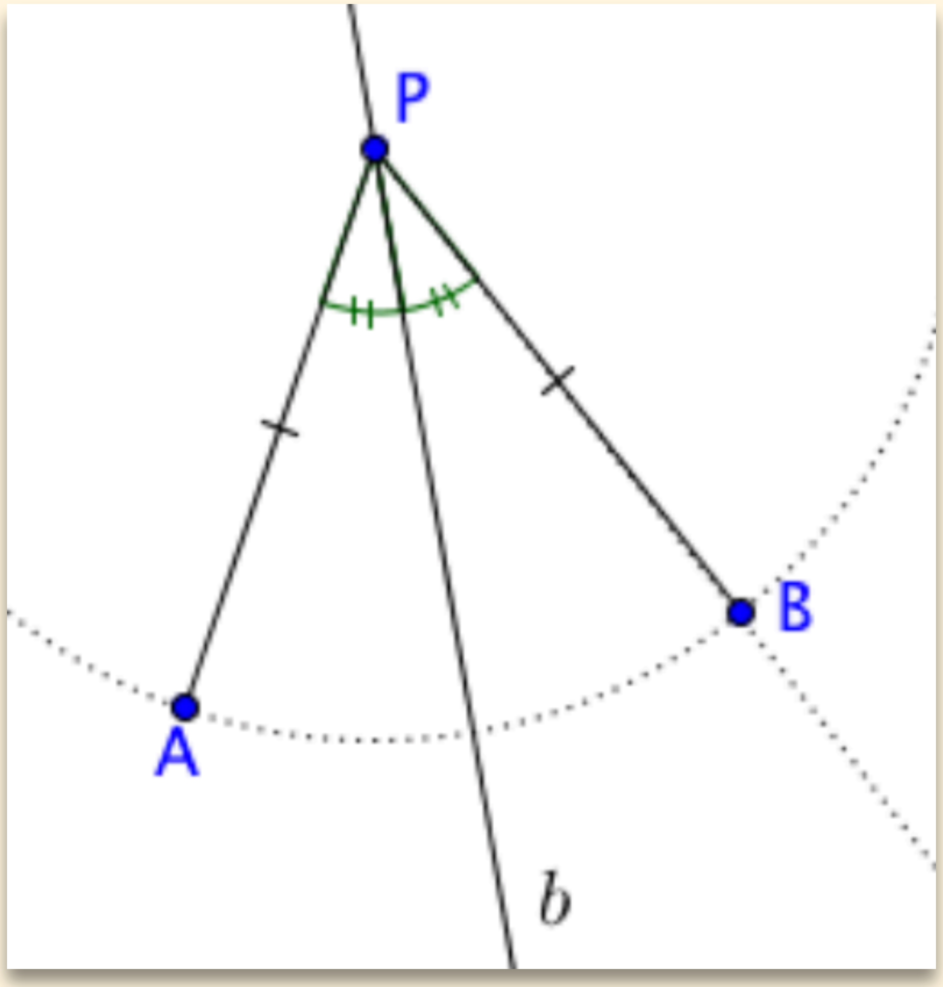
Construction Assumptions

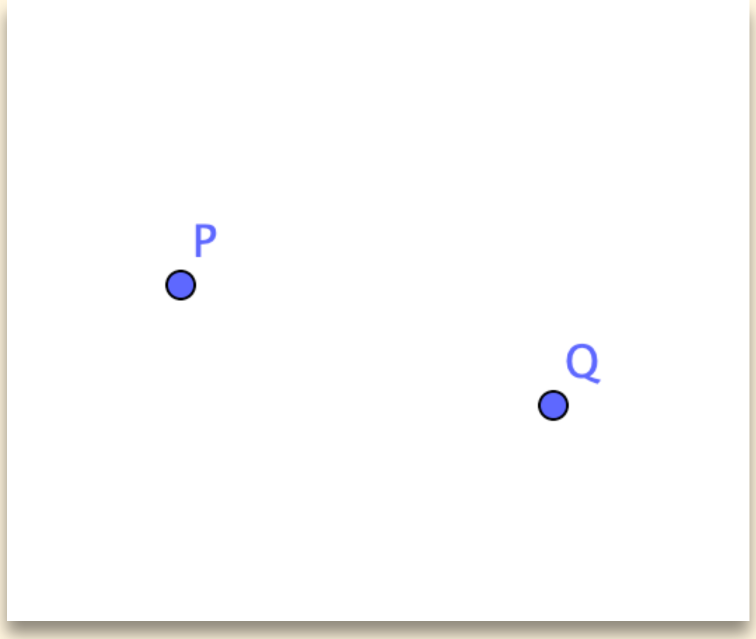
- ◊ two distinct lines meet in at most one point
- ◊ two distinct circles meet in at most two points
- ◊ a line and a circle meet in at most two points

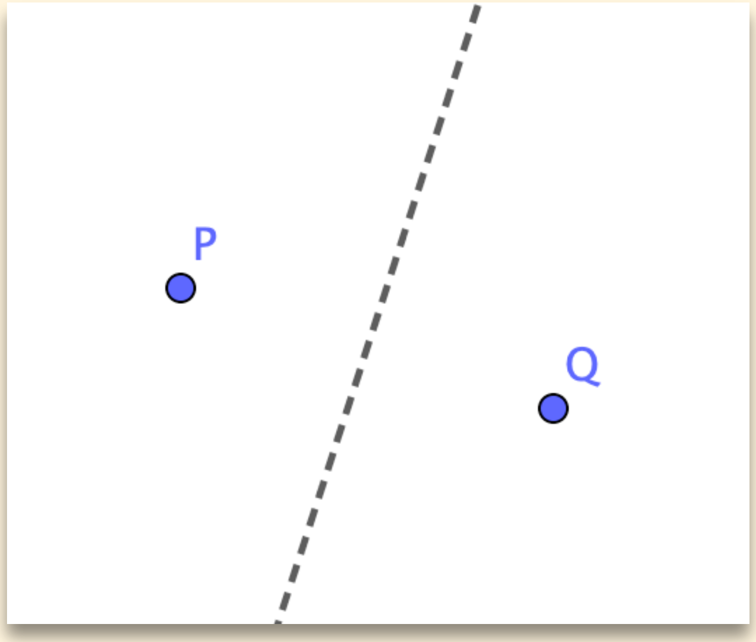
Perpendicular Bisector Theorem

A point P is equidistant from two points A and B if and only if it lies on their perpendicular bisector.

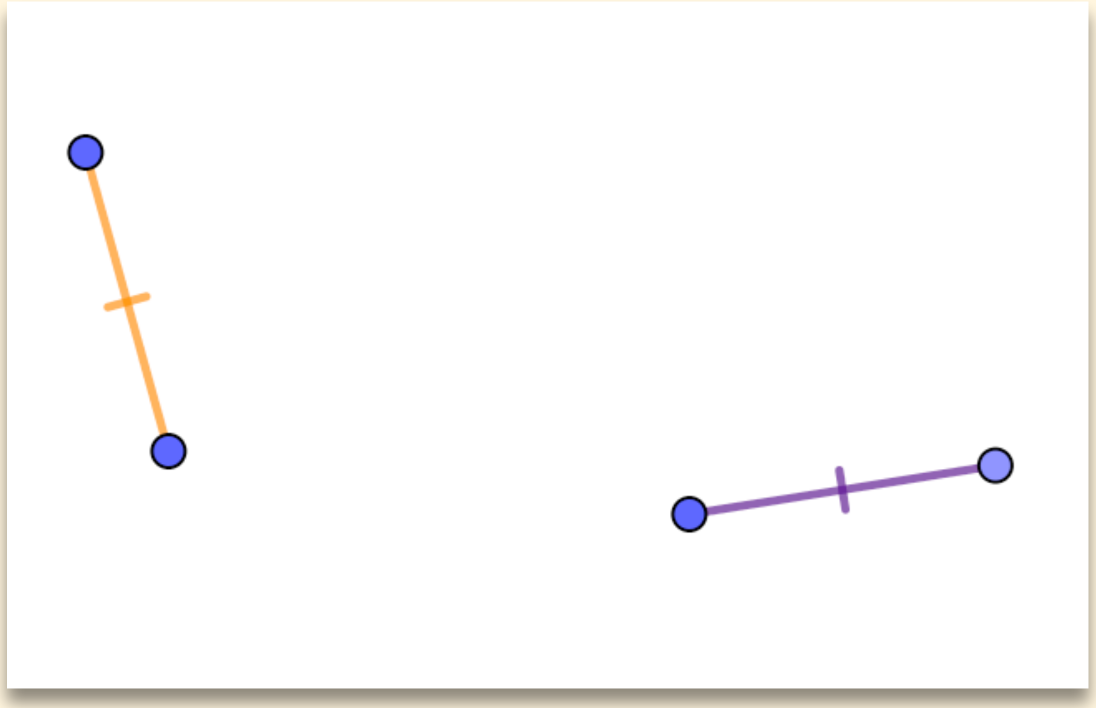


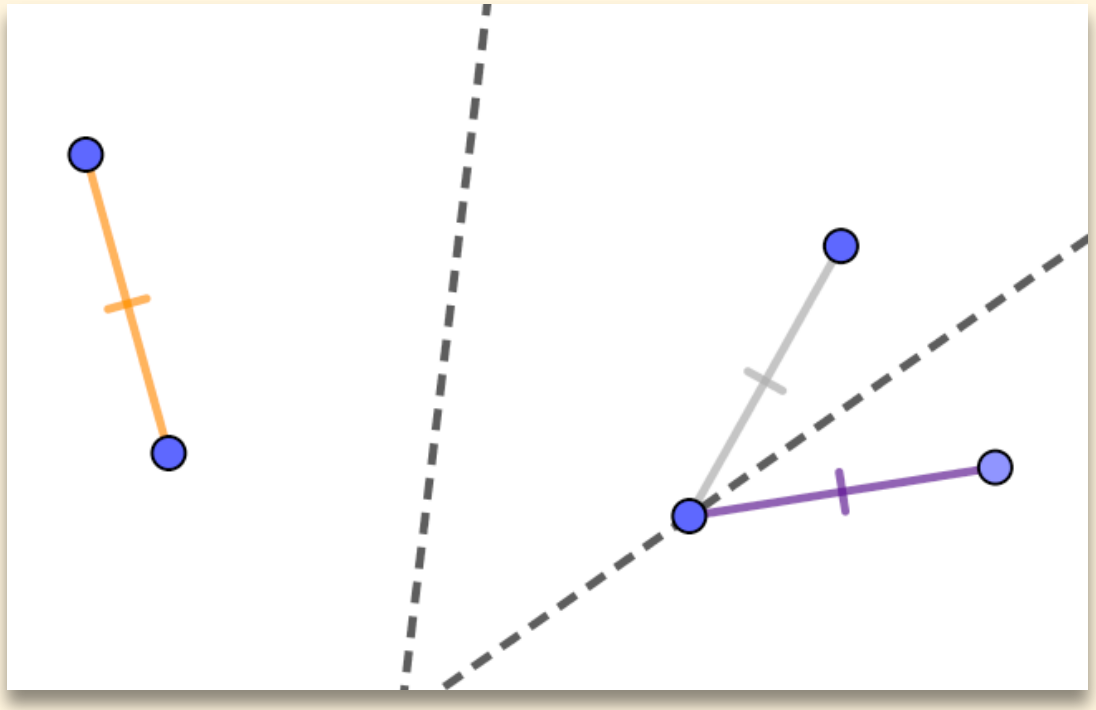






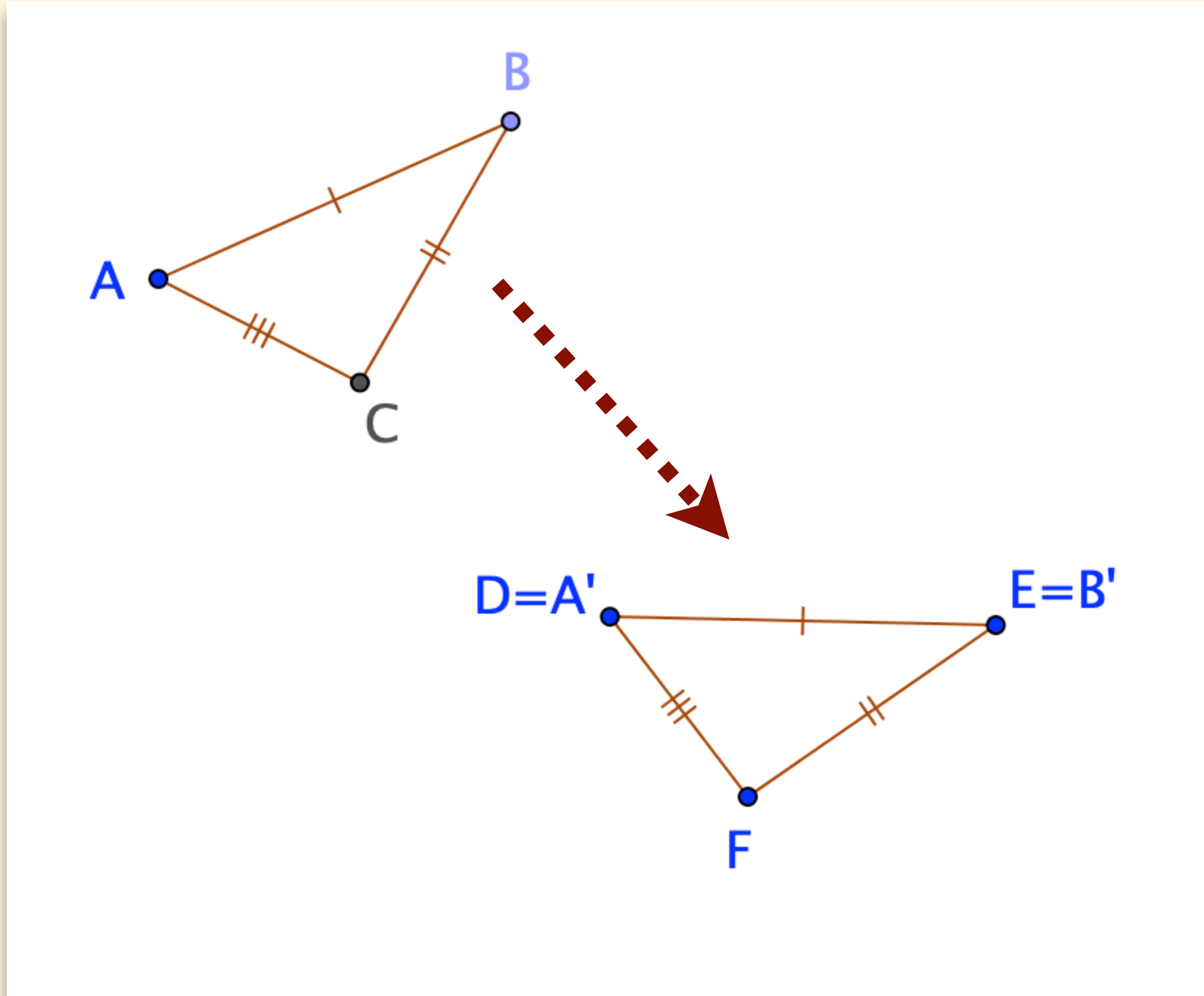
Theorem: There is a reflection that maps any given point P into any given point Q.

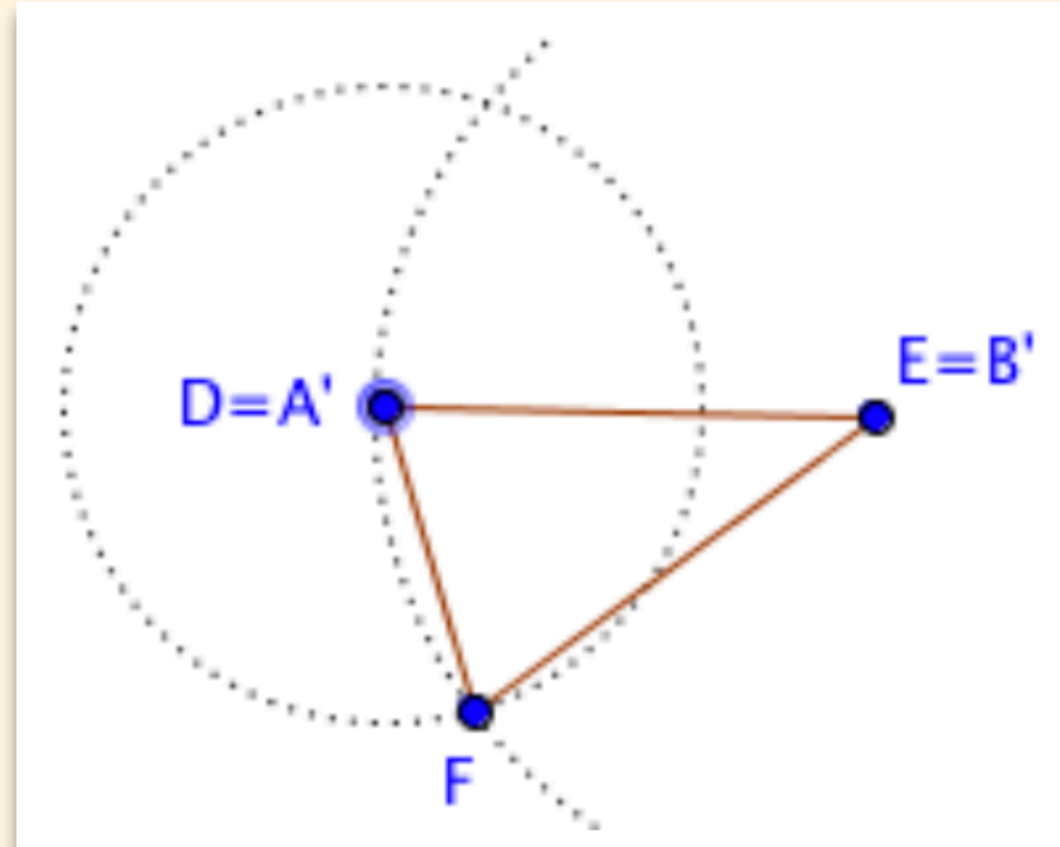




<u>Theorem</u>: If two segments have equal length, then one is the image of the other under either one or two reflections.

Segments are congruent if and only if they have equal length.





Triangle congruence: SSS

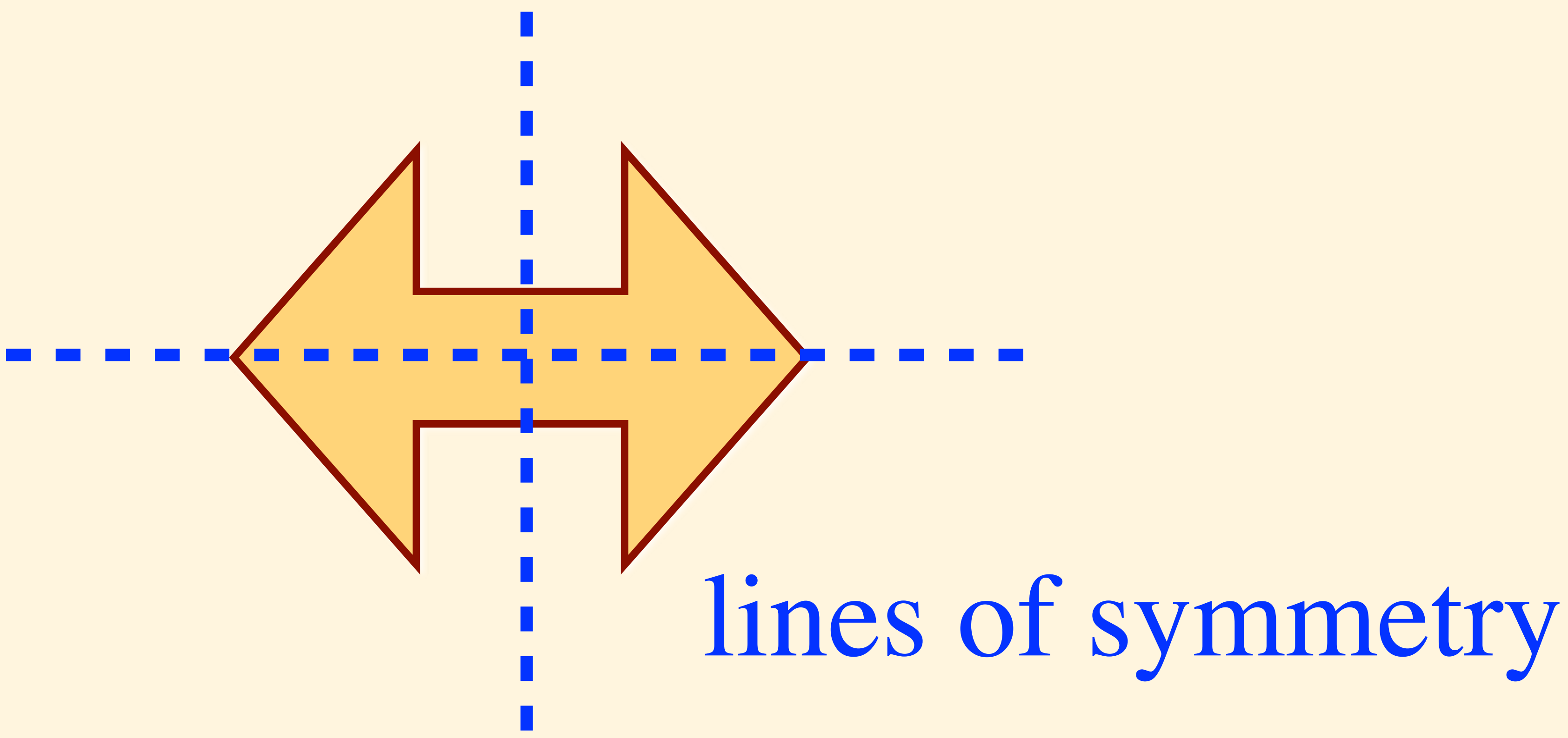
(SAS, ASA will work in similar ways)

We’ve filled the Common Core gap:
from basic principles to triangle congruence.

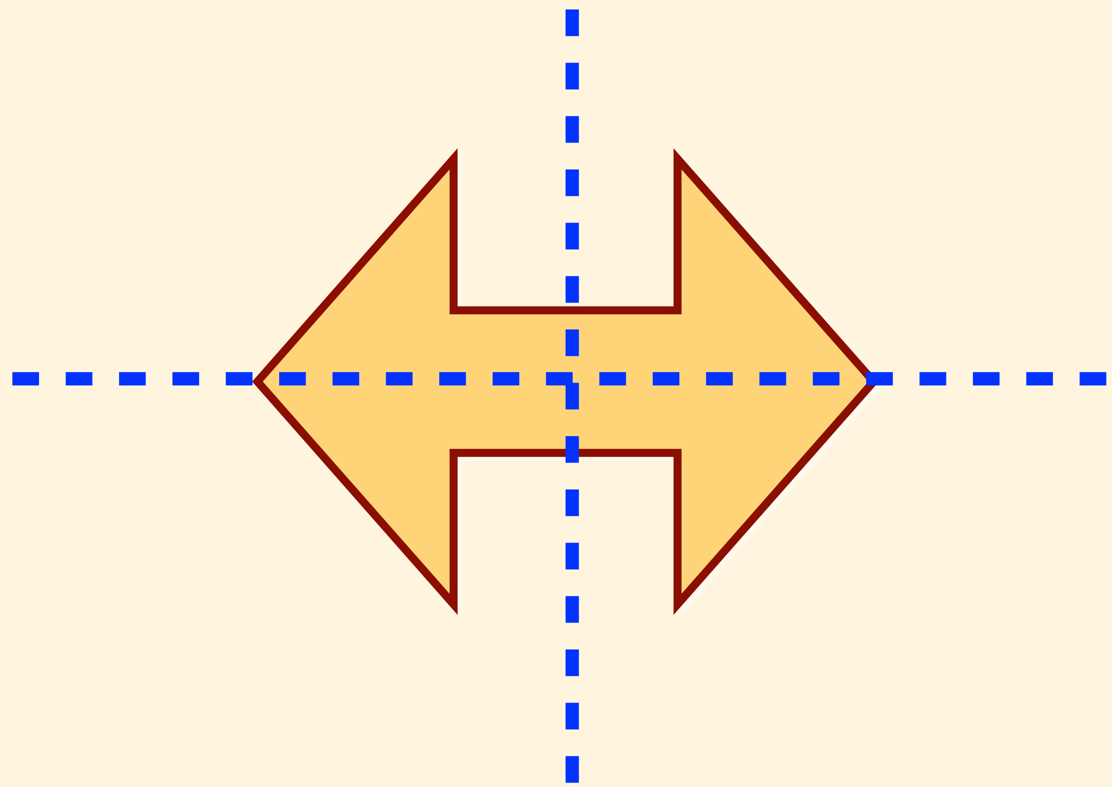
More Ambitious Content

Definition: A *symmetry* of a figure is an isometry for which the figure is invariant.

(The image is the pre-image. Individual points need not be fixed.)

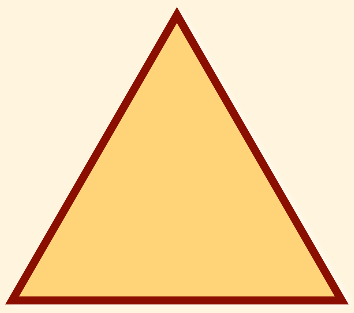


Two-fold rotational symmetry (180°)

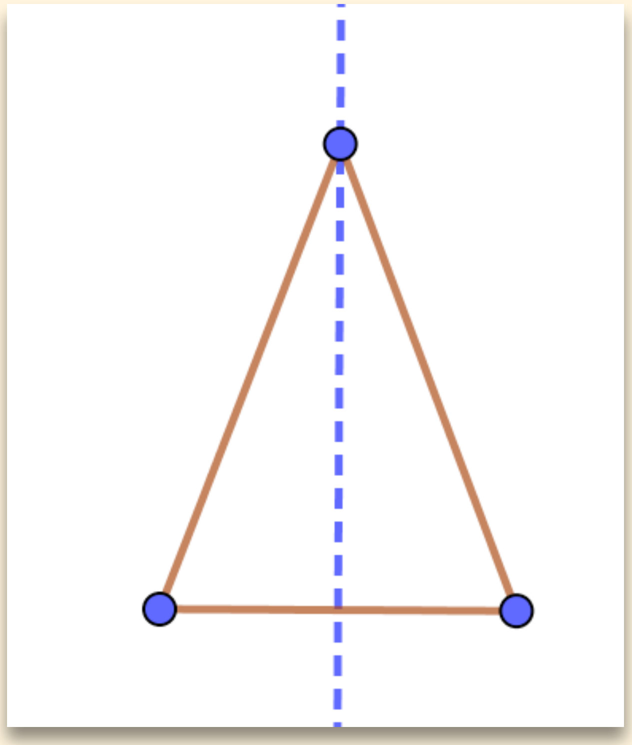


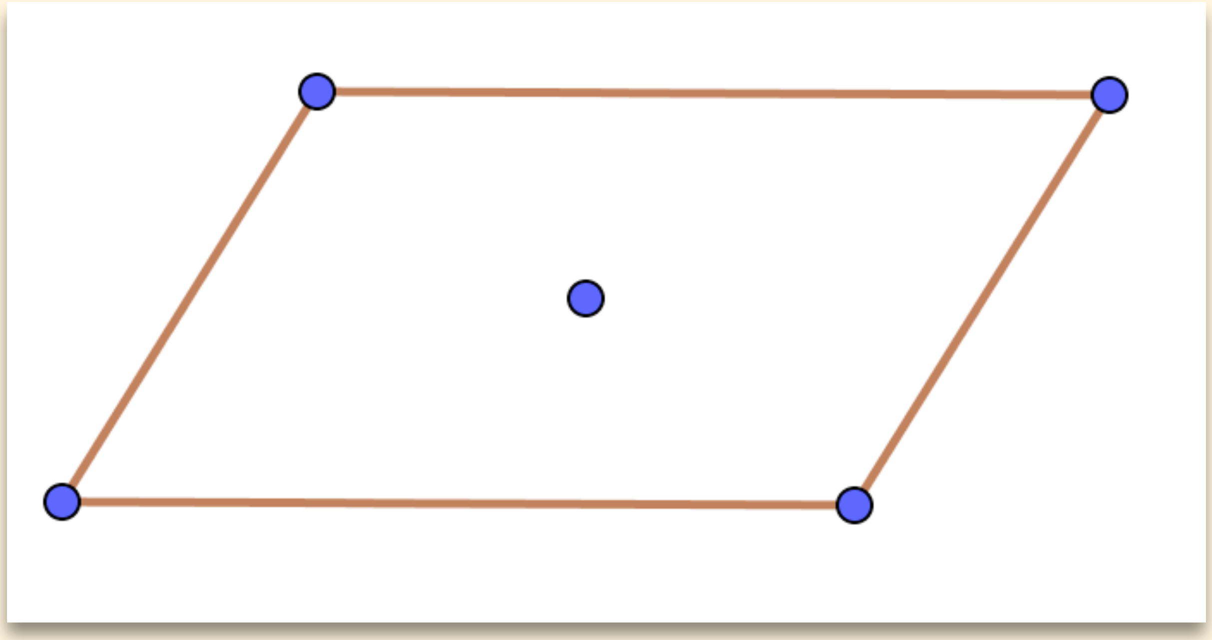
Z

Three-fold rotational symmetry (120°)



Symmetry definitions for special triangles and quadrilaterals





- ◊ Hierarchy
- ◊ Proving properties
- ◊ Proving a polygon is special

Transformational Proof beyond that

Not for everything, but useful for

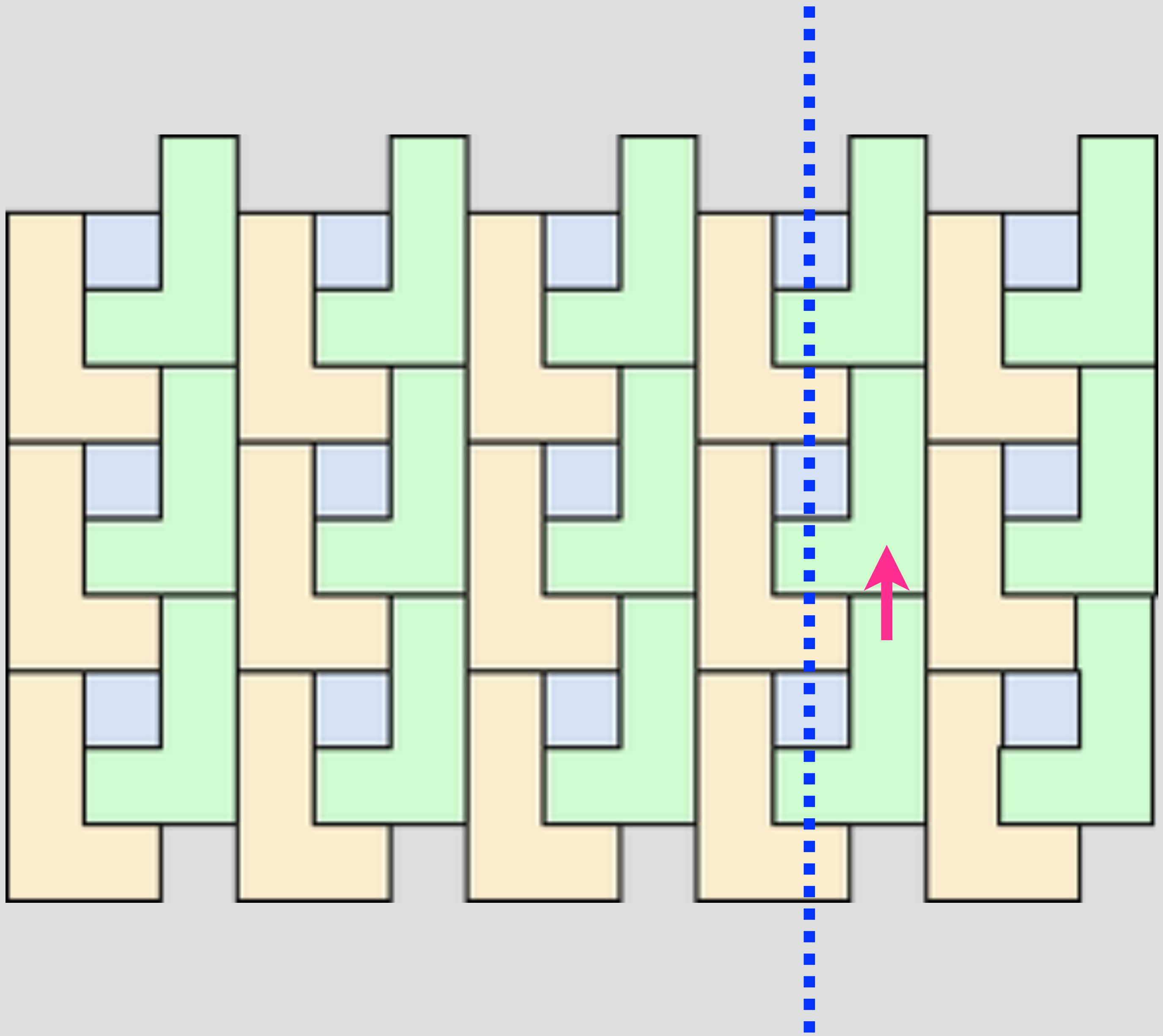
◊ Similarity

◊ Some theorems about circles

…and in subsequent courses

◊ all parabolas are similar
◊ computing images (complex numbers, matrices)
◊ glide reflection

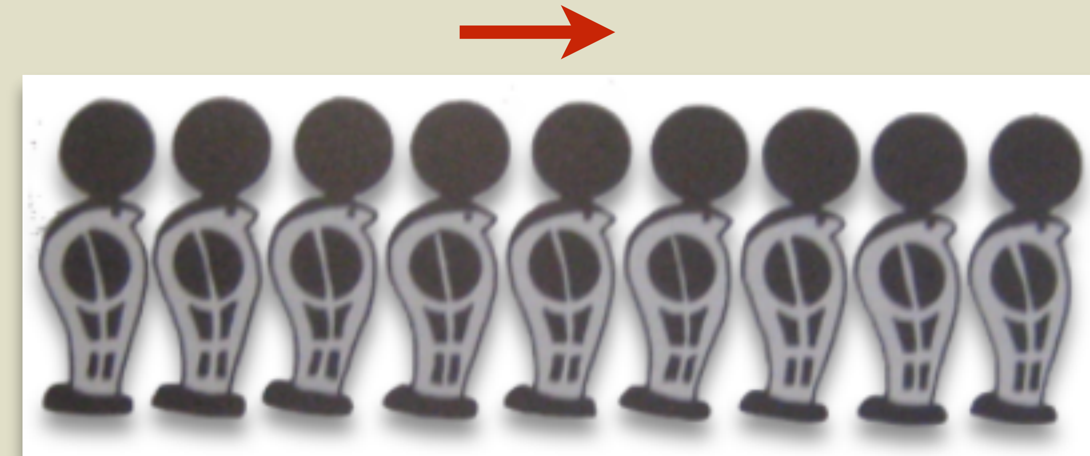
Glide Reflection



For more depth and enrichment

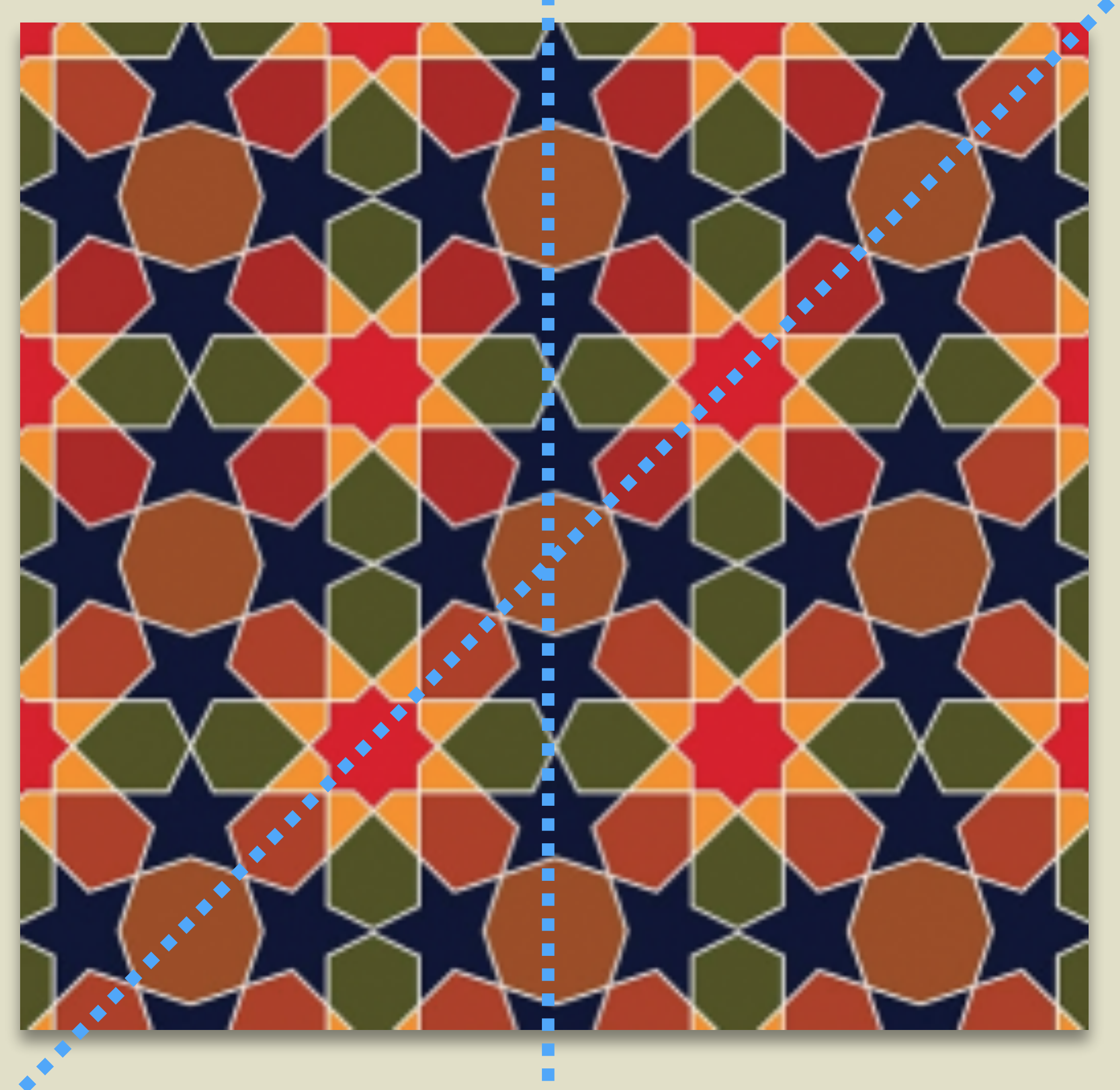
- ◊ only four types of isometries
- ◊ intro to abstract algebra
- ◊ intro to frieze and wallpaper symmetry

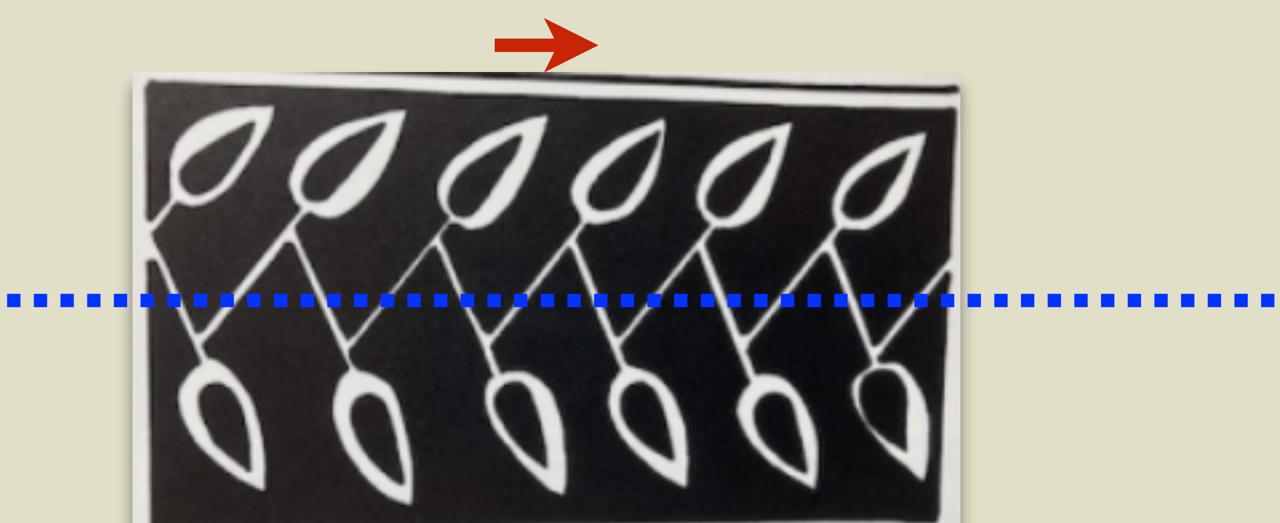
Escher



Egypt

Islam





India

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Henri Picciotto
www.MathEducation.page
henri@MathEducation.page
blog.MathEducation.page
Twitter: @hpicciotto