GUIDELINES FOR DISCOVERY TEACHING

A Handbook for Teachers of
General Mathematics and Prealgebra
in Grades 9-12.

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FOREWORD

Since 1963, mathematicians and scientists trained by Project SEED have been using a Socratic, group discovery method of instruction to teach abstract, conceptually-oriented mathematics (topics from high school and college level algebra) to elementary school children from educationally disadvantaged backgrounds. Not only were the students successful at learning algebra, but their basic arithmetic scores improved dramatically.

The basic Project SEED group discovery techniques were initiated by William F. Johntz, the founder and National Director of Project SEED. Over the years they have been refined by SEED specialists throughout the country into a powerful tool for teaching mathematics. Although the methods have been used primarily with elementary school students, they have implications for teaching students at all levels.

Project SEED techniques are designed to overcome the low motivation and feeling of academic inferiority that are common among students from low socioeconomic backgrounds. They are designed to maximize student involvement in the development of the mathematics being taught. Project SEED mathematicians concentrate on making students feel good about their ability to do mathematics and want to participate in each lesson. There also is an emphasis on developing the students' critical thinking and problem solving abilities.

During the school years 1980-81 and 1981-82, under contract with the U.S. Department of Education, Basic Skills Improvement Program, Project SEED staff in Atlanta, Boston, Chicago, Detroit, Los Angeles, and Oakland worked with ninth through twelfth grade students and their teachers in Title I eligible areas in an attempt to modify the SEED methods for use with secondary school students. Our work primarily focused on general mathematics* and prealgebra classes. Students in these classes usually have fallen behind the norm and seem to be trapped by their own feelings of academic inferiority. On the whole, our efforts were quite successful in providing these students with a successful learning experience. We remain, however, firmly convinced that it would be far more effective to work with these same students in elementary school, before their attitudes about their academic abilities have set so firmly.

The present manual, "Guidelines for Discovery Teaching," is the result of this experience with secondary school students. It is written for teachers of general mathematics and prealgebra classes, although it contains many suggestions that apply to other levels and other subject areas. It is designed to capture the essential elements of discovery teaching which we found successful with students in grades nine through twelve.

Usually mathematicians who are trained to teach by the Project SEED method learn their techniques through a procedure of observation, discussion, and supervised teaching over an introductory period of several weeks and continuously throughout their tenure as SEED specialists. Each mathematician who teaches SEED group discovery classes has learned the process through modeling and oral tradition. Experienced SEED specialists are successful because they have imitated a successful SEED specialist.

In using these notes to adopt discovery teaching in your classroom, you will not have this luxury. It is unlikely that there will be an experienced SEED specialist to observe or to give you feedback on your teaching. We recommend, therefore,

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*There are a variety of titles, such as "Math Workshop," "Consumer Math," and "General Mathematics," given to classes for secondary students who have not yet mastered the basic arithmetic skills to enroll in prealgebra or algebra. General Math is used throughout to refer to these classes.

that you begin by reading the entire manual quickly. This will give you an overall picture of the method that would have been provided by observation.

Begin discovery teaching by picking your own topic or using one of the Curriculum Modules developed by Project SEED. Go back through this manual and pick out one or two key techniques from each chapter to incorporate into your lesson. As they become part of your permanent repertoire, experiment with additional techniques. Use the checklist in the Appendix as a reminder to keep your methods varied.

We hope this manual will become a guidebook leading both you and your students to experience greater success.

CHAPTER 1

INTRODUCTION AND OVERVIEW

Introduction

If you were to observe a successful group discovery class, you would witness both the instructor and the students actively involved in the learning process. Students discover mathematical concepts through answering the instructor's questions. There is a positive atmosphere in which there are no "wrong" answers, and students who ask questions often receive more recognition than students who answer them. Students are confident of their ability to learn and frequently debate with the instructor or their peers.

To help orient the reader, we begin with an illustration. The following lesson is typical of hundreds of SEED lessons over the past nineteen years with students traditionally labeled "educationally disadvantaged" because of their socioeconomic and achievement levels.

The lesson began with some review material, which was designed to prepare a receptive mood for the central question "2E¯1=?" ("E" here is used for the operation of exponentiation so that 2E3=2x2x2=2^3.) The class had never considered questions on negative exponents before although, as shown by the review questions, they were familiar with the additive law for E and also with addition of integers.

When the instructor invited conjectures on 2E¯1 he received a number of answers, but most favored 2E¯1=1. The students' arguments in favor of this answer were various and interesting. The one which seemed to have the most support turned about the similarity between "2+¯1" and "2E¯1". Then one student suggested using "2E¯1 in a sentence: (2E¯1) x (2E2)=2E(¯1+2)=2E1. From this the class derived, in the usual way, that 2E¯1 was acting like $\frac{1}{2}$: $(2E^{-}1)\times(2E2)=2E1$

$$\begin{array}{ccccc} (2E^{-}1) & \times & (2E2) & = & 2E1 \\ \downarrow & & \downarrow & & \downarrow \\ \boxed{\frac{1}{2}} & \times & 4 & = & 2 \end{array}$$

This result provoked a truly excellent debate, with some students arguing for the previous answer ($2E^{-}1=1$), and other students for $2E^{-}1=\frac{1}{2}$. One student devised a further argument for $2E^{-}1=\frac{1}{2}$ by pointing out the pattern

$$\begin{array}{l} 2E3=8 \\ 2E2=4 \\ 2E1=2 \\ 2E0=1 \\ 2E^{-}1=\frac{1}{2} \end{array}$$

Throughout the lesson there was a great deal of student dialogue, which the instructor handled in a purely non-legislative fashion--being careful to keep open the status of $2E^{-}1$.

At one point in the discussion, a student was insisting that "2 and $^{-}1$ make 1, not $\frac{1}{2}$. And $\frac{1}{2}$ doesn't make sense." Whereupon another student replied that "2 and $^{-}1$ are 1 in <u>addition</u>, but not in exponentiation.

In the above lesson, the students were thinking critically about mathematics, exchanging ideas and listening to each others' opinions. There are numerous strategies and techniques which are used to establish this classroom situation.

The techniques fall roughly into four major categories which are described briefly below. The remaining chapters of these guidelines elaborate more fully on each of the categories, and contain detailed suggestions on how to successfully incorporate discovery teaching into the teaching of general mathematics and other subjects. Specific techniques often accomplish more than one purpose; however, to avoid repetition, we have tried to place each where it is used most frequently.

Key Elements of Discovery Teaching

There are four main elements of discovery teaching as used by Project SEED.

1. Questioning - Virtually all the dialogue in SEED classes consists of instructor questions and student responses, except, of course, when the instructor responds to a student's questions with another question. Structured sequences of questions lead the class to an understanding new mathematical concepts. Other questions review and reinforce previously learned material. Chapter 2 is devoted to general strategies for presenting material without lecturing.

2. Advanced, conceptual mathematics - Students often view mathematics as a sequence of arbitrary, meaningless rules to be memorized. It's no wonder they often don't retain what they learn. Conceptual understanding is an essential component of discovery teaching. Students who thoroughly investigate and understand key examples are better able to retain what they've learned and apply it to new situations.

   Frequent repetition of material leads to boredom, resentment, and perhaps discipline problems, even when students have not mastered the material. This is particularly true of general mathematics students, who know that the material they are covering is normally taught in 6th or 7th grade. SEED instructors choose advanced topics which embed and reinforce elementary ones (or teach elementary concepts from an abstract, conceptual point of view). Because the material is fresh, they are able to capture the students' interest, while at the same time reinforcing basic skills.

Chapter 3 contains techniques for structuring questions to develop conceptual understanding. The reader is also referred to the curriculum modules which are a companion to these guidelines. The modules contain outlines illustrating in detail the Socratic development of selected topics.

3. Feedback and involvement - The instructor constantly uses a variety of techniques to monitor student understanding and to maintain a high level of participation. The teacher uses information about the class level of understanding in formulating the next question. Feedback and involvement techniques are found in Chapter 4.

4. Positive, supportive atmosphere - The previous three characteristics are combined with additional techniques for supporting and motivating students to present an atmosphere in which there is a strong individual and group sense of achievement and success. There are no wrong answers. Students feel that they can learn and they do. Chapter 5 describes techniques for building student confidence.

A Note on Large Group Instruction

The techniques contained in this manual deal primarily with whole group instruction. While individual practice (seatwork and homework) are clearly necessary to reinforce skills, we have found that the discovery process for learning mathematics works most effectively with an entire class, particularly in courses like general mathematics, where the students lack confidence in their own academic abilities.

Some reasons for this somewhat surprising phenomenon are:

1. <u>Frees the student to think</u>.

A group of sufficient size (20-30 students) can create an illusion of anonymity for the student, freeing him or her* to think creatively. In a one-to-one relationship with the teacher, or in a small group of three or four, the student is put on the spot when he is asked a question that calls upon him to think or reason. Most people tend to freeze--or at least to become anxious--when asked questions that require them to think. To be singled out can only intensify the student's anxiety. The large group enables him to feel less exposed, less vulnerable. The sensitive teacher can help an insecure student relax in a large group: He can supportively "ignore" him for a few minutes by turning his attention to other students giving him the time he needs to unfreeze and think through a question, a procedure that would obviously be impossible in a one-to-one or small group situation, since ignoring the student under those circumstances, however benign it might be, would be obvious to the point of seeming accusatory.

2. <u>Students more likely to perform</u>.

By the same token, the group situation, by providing the student with a sense of anonymity, tends to destroy in him any lingering passivity. Students can be asked to respond in chorus and to use hand symbols and gestures in large groups. Such requests would seem foolish in a small group or the one-to-one relationship, merely

*There is no acceptable convention for dealing with the third person singular. In these notes, we occasionally use the awkward "he or she," but generally arbitrarily choose one gender or the other.

causing the student to feel embarrassed. Moreover, there are ways in which the skillful teacher can draw out a truly reluctant student in a group situation that would be impossible if she were alone with him. (See Chapter 5)

3. <u>Student's self-concept enhanced</u>.

   When she responds in a group situation, the student's self-concept is much more likely to be enhanced than when she responds individually to the teacher. First of all, the group situation makes it possible for the teacher to use subtle, indirect methods of involving the student in the learning process. Once involved, the student feels better about herself, and her self-concept inevitably is enhanced as a result.

   Second, students are far more eager for peer approval than for teacher approval. The skillful teacher who provides students with an exciting group learning experience, one in which becoming involved is "the thing to do," creates in the reluctant learner a need to participate in order to seek peer approval. Once he knows he has earned peer approval by performing well, his self-concept has been even further enhanced and he is motivated to perform even better.

4. <u>Group discussions are more productive</u>.

   The richness of a discussion among students in a large group is far more productive than what is ever possible in a small group or between the teacher and one student, simply because there are more people contributing their ideas. When students discuss particularly difficult subjects, such as mathematics, a group can often solve together a problem that no single one of them could solve alone. Such a shared experience of success by a group adds a dimension to learning impossible to duplicate in the small group or one-to-one setting.

5. Students like group learning.

Watch students in a well-taught group learning situation. They learn and interact with zest and enthusiasm impossible to duplicate in an individualized setting. Students like to compete with their peers; they become truly involved in the content of the discussion and will often spend time outside of class "researching" a particular point in order to contribute to a later discussion.

CHAPTER 2

GENERAL SOCRATIC STRATEGIES

Effective Questioning Techniques

Good questions are at the heart of the discovery method. Carefully sequenced sets of questions enable students to understand extremely sophisticated material. The most successful discovery teachers only ask questions and rarely or never make declaratory statements in class about anything more serious than the weather.

Questions allow the instructor to gain immediate feedback from the students and, thus, to pace the introduction of new material appropriately. Students become active rather than passive participants in the learning process. They are focused on the topic at hand and begin to develop a framework for problem solving on their own.

The following suggestions will help you design question sequences that stimulate true discovery and critical thinking in a group setting.

1. Write out your sequence of questions in preparation for the class. This will help you find some of the areas of difficulty. You can then write a sub-routine of questions that will help ease the students over the difficult spots. A useful exercise is to consider what questions you would need to ask if you, yourself, were learning the topic.

2. Keep a log. Because discovery teaching moves with the response of the class, it is imperative to keep a record of the day's lesson. Many instructors have a student take notes. Others spend a few minutes annotating the day's plan after class and use that as a springboard for planning the next day's lesson.

3. Vary the difficulty of questions. In order to keep students of different abilities constantly involved in the lesson, you must ask a mixture of hard and easy questions. Challenging questions keep the faster students interested, while routine questions give the slower students a chance to respond and build confidence.

Examples:

Problem	Routine Questions	Challenging Questions
$\frac{1}{2} + \frac{1}{4} = ?$	Who can find an equivalent fraction for $\frac{1}{2}$ that we can add to $\frac{1}{4}$? What's $\frac{2}{4} + \frac{1}{4}$? Who can draw a picture to illustrate this problem?	Who can find two different fractions whose sum is $\frac{1}{2}$? (Is $\frac{3}{6} + \frac{2}{8}$ a legitimate answer?)
$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$	What's the next question?	What problem would we solve to find S_{10}?
$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$	What problem do I have to solve to find S_4? (All the steps to computing $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$) Who can predict S_5?	Find a formula for S_n.

3. Vary the question pace. A varied pace helps avoid monotony and maintains student interest. Pace will also depend on the degree of difficulty of the question being asked.

Alternate fast-paced series of straightforward questions with slower conceptual discussions. Most discovery classes begin and end with a quick pace. Fast-paced questions are also useful when reviewing, building up to a generalization through examples, or practicing an idea the class has just verbalized. Have the class chorus their answers when you really want to move fast. Using quick consecutive chorus responses is a very effective device for focusing a class. Appropriate times for a more relaxed pace include: when the class is discovering a key concept, when students are attempting to articulate thoughts, when they are checking a result on their papers, and when a student is teaching.

5. Vary the mode of response. This again will avoid monotony while involving more students. In addition to calling on an individual, you can: ask the whole class to say an answer, ask a particular group of students to say it, have the students write the answer on their papers, or have them use their hands to express an answer or to signal agreement or disagreement with someone else's answer. In order to avoid confusion and called out answers, it is advisable to make it clear to the class, when you ask the question, how you want them to respond.

   A typical pattern is to ask four or five similar questions in a row. Call on individuals for the first two. Then have the whole class answer a question in chorus. Finally, have each student work a problem on paper. You might also have a student work the problem on the board. Ask about half your questions so that they are answered by the whole class together in some way, and the other half of your questions so that they can be answered by individuals. Make sure that each member of your class has a chance to answer several questions each day.

   (See also Chapter 4 on Feedback and Involvement)

6. <u>Ask a variety of questions on the same concept</u>. Asking a variety of questions on the same concept gives more students a chance to participate, reinforces skills, and may give students new insights and understanding. Often students who didn't understand a concept the first time find it makes sense when presented from a different angle.

7. <u>Set up patterns</u>. Patterns help the class move smoothly from simple to more complex examples. Patterns can also be used to provide a challenge (discovering the pattern) to students who already understand the concept. Less complex patterns can be used to provide questions (predicting the next problem) that less advanced students can answer successfully. Since patterns can sometimes be misleading, it is wise not to always be predictable. (For a more thorough description of patterns, see Chapter 3.)

Examples:

$(\frac{1}{2} + \frac{1}{2}) \div \frac{1}{2} = ?$, $(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \div \frac{1}{2} = ?$,

$(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}) \div \frac{1}{2} = ?, \ldots;$ $\frac{1}{3} + \frac{1}{3} = ?$, $\frac{1}{3} + \frac{1}{6} = ?$,

$\frac{1}{3} + \frac{1}{9} = ?$, $\frac{1}{3} + \frac{1}{12} = ?$, . . .

$\frac{1}{2} + \frac{1}{3} = ?$, $\frac{1}{3} + \frac{1}{4} = ?$, $\frac{1}{4} + \frac{1}{5} = ?$ . . .

$\frac{1}{2} \times 4 = ?$, $\frac{1}{2} \times 6 = ?$, $\frac{1}{2} \times 8 = ?$, . . .

$2^1 = ?$, $2^2 = ?$, $2^3 = ?$, . . .

Note that most of the sequences just given admit of generalizations:

$$\sum_{i=1}^{2n}\left(\frac{i}{i} \cdot \frac{1}{2}\right) \div \frac{1}{2} = n, \quad \frac{1}{3} + \frac{1}{3 \cdot n} = \frac{n+1}{3n}, \quad \frac{1}{n} + \frac{1}{n+1} = \frac{2n+1}{n(n+1)}, \quad \frac{1}{2} \times 2n = n.$$

In teaching such patterns, it is often a good idea to arrive eventually at the general form, provided it can be done, which it usually can, without undue strain. Hence one might ask "$\frac{1}{3} + \frac{1}{3.\lambda} = ?$" or--more difficult--who can generalize this?

8. Use parallel problems for contrast.

$\frac{1}{2} + \frac{1}{3} = ?$	versus	$\frac{1}{2} \times \frac{1}{3} = ?$
$10 \times \frac{1}{2} = ?$	versus	$10 \div \frac{1}{2} = ?$
$\frac{1}{10} \div \frac{1}{10} = ?$	versus	$\frac{1}{10} - \frac{1}{10} = ?$
$5 + ? = 0$	versus	$5 \times ? = 1$

These kinds of problems are also helpful in reviewing and consolidating. For instance, one might start out the class by writing the following sentences on the board:

$\frac{1}{2} + \frac{1}{3} = \underline{\ 5\ }$	$\frac{1}{2} \times \frac{1}{3} = \underline{\frac{1}{5}}$
$10 \times \frac{1}{2} = \underline{\ {}^{-}5\ }$	$10 \div \frac{1}{2} = \underline{\ 1\ }$
$\frac{1}{10} \div \frac{1}{10} = \underline{\frac{5}{6}}$	$\frac{1}{10} - \frac{1}{10} = \underline{\frac{1}{6}}$
$5 + \underline{\ 0\ } = 0$	$5 \times \underline{\ 20\ } = 1$

Then the class is asked to permute the solutions so as to make all sentences true.

9. Be specific. Unless you want to generate an open discussion, make your questions specific. A question that can be answered quickly frees you to move on to

the next question. Specific questions also avoid correct answers that slow you down or lead in a different direction from what you had planned.

Examples:

$.7 \times .05 = .035$	If you want to review the relationship of fractions and decimals, ask "How can we check this answer using fractions?", not "Why is this the right answer?"
$(2 \times 3) + (2 \times 4) = ?$	If you are trying to teach the distributive law, ask for an answer that uses the numbers in the problem.

10. <u>Leave some questions open until the next class or even longer</u>. It creates an intellectual challenge which students may think about on their own. It also gives you a chance to present more tools or a different approach for solving the problem so that more students will grasp it. When the class eventually solves the problem, there is a greater sense of accomplishment--"This problem was so hard it took us three days to solve it."

    Many instructors have students keep a page of "Unsolved Problems" in their notebooks. These are frequently questions like $3 - 5 = \boxed{}$ that arise during class and will be covered later in the year. Special recognition should be given to students who solve problems from the list or who bring them to the class's attention when it is ready to solve them.

11. <u>Don't push for results the class isn't ready for</u>. If the students are forced to a conclusion they don't quite understand, they lose confidence and are likely to forget what you've taught. It is preferable to return to the subject at the next lesson, perhaps approaching it from a different direction.

The same principle holds true for verbalizations and generalizations. When a new concept is introduced, students frequently discover how to solve problems without being able to articulate how they got their answers. Forcing them to verbalize prematurely may lead to confusion. If only a few members of the class can provide an explanation or generalization, the rest of the class tends to rely on rote memorization of their classmate's rules, rather than their own conceptual understanding.

12. <u>When a student asks a question, don't answer it yourself</u>. Ask for another student to volunteer to answer it, or ask a sequence of questions that will lead the class to answer the question. Praise students who ask questions to encourage more questions.

13. <u>Don't kid yourself</u>. Making a declarative statement and following it with the question, "Right?" is not using the discovery method. Such questions are really lectures in disguise, since the students will give the obvious answer, "Yes," without understanding the concept at all.

Review

Review is a key factor in successful discovery teaching. Review doesn't have to be done at the beginning of the lesson, although frequently the instructor who begins a class where it left off at the last lesson finds the students getting bogged down in material they previously handled easily. A brief <u>conceptual review</u> should be part of each class. This reinforces the concepts and allows the students to build on them later. It also provides a success experience for the students. Review enables new students, absentees, and slower students to catch up to the rest of the class, and more advanced students to gain new insights and deeper conceptual understanding. Review provides a springboard for new material.

In doing review, the overall strategy is to disguise it and keep it fast paced and surprising. In general, review questions should be skew-directional, the next question in the sequence being unpredictable. Occasionally, however, predictability itself can be a virtue. A question which you can then use with the class is, "What do you think I'm going to ask next?" And, of course, the spiral method should be employed regularly--asking questions on the same material but at a different level.

Review should focus both on conceptual understanding and fast-paced drill. During the course of a two-week period, the entire year's work should be touched on to enable the students to maintain the mastery of a considerable body of mathematics. This gives them a sense of intellectual strength and increases their willingness to risk the unknown. On the average, 20%-60% of any lesson will be review.

There are a number of ways of making the review portion of the lesson as interesting as the discovery of new concepts. Some suggestions follow:

1. Cover familiar ground rapidly. For example, you might lead up to a generalization with only one or two examples, or ask a series of rapid oral questions.

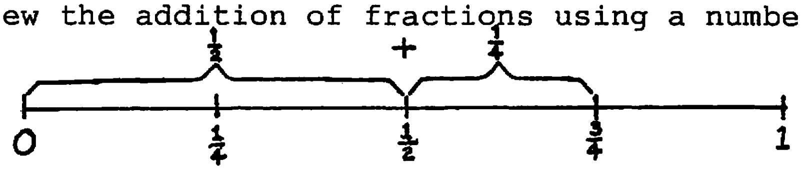
2. Approach a review concept from a new direction. Examples:

Approach $2^0 = 1$ via division $2^0 = \frac{2^3}{2^3} = 1$

or by the pattern

$$\begin{aligned} 2^3 &= 8 \\ 2^2 &= 4 \quad \div 2 \\ 2^1 &= 2 \quad \div 2 \\ 2^0 &= 1 \quad \div 2 \end{aligned}$$

Review the addition of fractions using a number line.



3. Try to ask more provocative questions that lead the students deeper into the mathematical concepts.

Examples:

<u>Inverses</u> - What are the similarities between
$^{-}2 + 6 = \square$ and $\frac{1}{2} \times 6 = \triangle$?

<u>Decimals</u> - If $23296 \div 728 = 32$, what's $23.296 \div 3.2$?
(Have students do the second problem in their heads to reinforce the rule for placing the decimal point when multiplying decimals.)

<u>Open sentences</u> - Who can make up an open sentence that has 2 in its truth set?

Who can make up an open sentence that has an empty truth set?

4. Choose new material that presupposes previous concepts so that review is an essential part of the lesson.

Example: Review the additive law of exponents ($a^n \times b^m = a^{(n+m)}$) and multiplicative inverses ($a \times \frac{1}{a} = 1$ for $a \neq 0$) in showing that $2^{-1} = \frac{1}{2}$.

5. Embed review material in new concepts.

Example: $(\frac{1}{4})^3 = ?$

$2^{3.16} \times 2^{4.7} = 2^{?}$

6. Introduce new notation or terminology.

   Example: In reviewing multiplication of fractions, introduce the words "multiplicative inverse" and "reciprocal."

   Change the notation for exponentiation from 2E3 = 2 x 2 x 2 to the standard $2^3 = 8$. Introduce the words "base" and "exponent."

7. Have students make up review questions. Be specific about the directions, such as: "Who can make up a problem using two to the zero power?" so that the review will keep the focus you want.

8. Put on the board a number of mathematical statements containing errors. Ignore the students' disagreement until you've finished writing. Feign disbelief in your mistakes, reluctantly admitting the existence of further errors only after the class corrects you.

9. Mid-lesson reviews can provide challenge and variety if you "mistakenly" or deliberately erase all or part of the board and then ask the class to help you put it up again. This is useful when you want to use previous results but need to display them differently.

10. As you are winding up a lesson, have students make up review questions for you to use the next day. Place students in charge of remembering particular results. An interesting variation is to ask the students to scan the whole blackboard and suggest which sentences can be erased and which should be kept for tomorrow, or which best capsulize what the day covered.

11. Use worksheets, letting students ask each other for help or refer to their notebooks.

12. Occasionally inject a review question into the middle of a lesson as a change of pace.

13. Use any of the various techniques listed elsewhere which increase student involvement and participation.

Vocabulary

In discovery teaching, vocabulary is introduced in context.

Examples: 1) $2^3 = 2 \times 2 \times 2 = 8$

$2^4 =$ "What if I change the <u>exponent</u> to 4?"

2) "Who can give me a division sentence that undoes $4 \times 8 = 32$? Who can give me a division sentence <u>equivalent</u> to $4 \times 8 = 32$?"

Answer: $32 \div 4 = 8$

$32 \div 8 = 4$

New words are written on the board either off to one side or linked to an appropriate example.

Example: exponent

$2^{3} = 8$

Students should keep a vocabulary page in their notebooks and should be reminded to record new words as they are encountered. The list should occasionally include nonmathematical words unfamiliar to the class that arise during the course of a lesson.

Vocabulary should be part of the ongoing review. The instructor should ask questions using the vocabulary to verify student understanding. Students might also be asked to provide or circle on the board an illustration of the word being reviewed. Key vocabulary words should be written on the board frequently, and students might be asked to read or spell the more difficult ones.

Boardwork

A clear, well-arranged board can facilitate the smooth flow of mathematical discovery. A useful planning device is to take a page and sketch exactly how you imagine the board will look at the end of the lesson. This helps to organize the space so that important hints and prerequisites will be left up and ideas don't run into each other haphazardly.

For example, the chart

$$
\begin{aligned}
2^4 &= 2 \times 2 \times 2 \times 2 = 16 \\
2^3 &= 2 \times 2 \times 2 = 8 \\
2^2 &= 2 \times 2 = 4 \\
2^1 &= 2
\end{aligned}
$$

leads to the obvious question, "What's 2^0?"

A useful technique for focusing students on important intermediate results is to enlist their help in deciding what can be erased and what should be saved. You might ask the class to read certain important results on the board while you write them off to one side to make space for new work.

Another helpful technique is "circling" for emphasis or focus.

Examples:

a) $^{-}2 + 2 + 5 = \square$ "Who can draw a closed curve around the part that adds up to zero?"

b) $(2E3)E4 = (2E3) \times (2E3) \times (2E3) \times (2E3)$

$= 2E(3 + 3 + 3 + 3) = 2E(3 \times 4)$

Who can read the part of the sentence in the closed curve?"

c) 2E3 = 2 x 2 x 2 "Who can circle the base?"

d) $\boxed{\sum_{i=1}^{5}}$ i — $\boxed{\text{The summation}}$ from i=1 up to 5 of i , etc.

CHAPTER 3

MATHEMATICS DEVELOPMENT STRUCTURES

The goal in discovery teaching is for students to feel that the mathematics they learn belongs to them. They understand it. They developed its rules and formulas. They can explain it to someone else. They are confident about their ability to extend their knowledge.

The basic process to accomplish this goal begins with problems the students understand or can solve readily, and builds more and more complex concepts upon them. The trick is to structure the sequence of questions so that the mathematical concept being taught unfolds as the students answer them. There are several recurring question patterns or structures which we have found useful in leading students to mathematical discoveries with conceptual understanding. They also help develop the students' critical thinking skill and provide them with a framework for solving problems on their own.

Gradual Escalation to a Generalization

Generalizations are central to mathematics and to the development of mathematical concepts in discovery classes. Students can discover many general mathematical principles or laws by investigating a number of particular examples. Variables can then be used to write a formula which expresses a true statement for all members of a given set. For example, the statement "zero times any number is zero" becomes "$\forall \alpha$, $0 \times \alpha = 0$."

The most basic method for arriving at a generalization and understanding its power is usually referred to as "erase and replace." A description of the process, along with an example, follows:

(a) Begin with several particular examples the class can solve.

$$3 + \boxed{^{-}3} = 0$$

$$5 + \triangle = 0$$

(b) Change the term you want to generalize--"If I change the 5 to a 17, what else has to change? (Note that there will be several correct ansers. Acknowledge each and continue until you get 17 + $^{-}$17 = 0.)

$$17 + \underset{^{-}5}{\triangle} = 0$$

(c) Continue to change the same term until the class realizes that any number can be used to make a true statement.

"Who can give me another number to put in this sentence?"

"Can I make a true sentence using 117?"

"Can I use a large number?"

"Can I use a small number?"

"Can I use a fraction?"

"How long can we go on putting different numbers here?

"How many numbers can we use?"

(d) Use variables to stand for arbitrary numbers. (Students seem to enjoy using Greek letters to stand for universal variables.) (Note: Technically, there should be a quantifier, "for every number γ " written " $\forall\gamma$ ". It is frequently omitted, which rarely causes confusion.)

$$\gamma + {}^{-}\gamma = 0$$

(e) Check by substituting particular values.

"Will I get a true sentence if I <u>substitute</u> a number for γ ?"

"Who knows what a substitute is?"

"What would it mean to substitute a number for γ ?"

"Who has a number to substitute for γ ?" Let $\gamma = 10$

"Who can read the sentence substituting 10 for γ ?" $10 + {}^{-}10 = 0$

"Is the sentence true when we substitute 10 for γ ?"

Another way to introduce a generalization is by doing several examples which illustrate the principle. Then ask a similar question involving variables, e.g.:

$$2 + \square = 0,\ 7 + \square = 0,\ 17 + \square = 0,\ \alpha + \square = 0$$

Note that this approach does not lead to an understanding of what α represents. However, it is an efficient method for arriving at generalizations once the class understands what they are. To reinforce the concept, it is important to ask review questions such as "What does α stand for?", and "Will I get a true sentence if I substitute numbers for α?"

Generalizations are often a good way to help demystify mathematical symbolism and encourage students to try problems, whether they think they know how to do them or not.

Examples: $f'(x) + \square = 0$

$$\int_0^5 x^2dx + {}^{-}\int_0^5 x^2dx = \square$$

Introduce the words "generalization" and "generalize" in context at some point by asking, "How would a mathematician show that we can use any number; how would she generalize this statement?" As the students become more familiar with the process, you can leave out more and more steps, giving only a few examples or saying, simply, "Who can generalize this statement?"

Be careful, however, about pushing the class to arrive at a generalization through a mechanical process before they thoroughly understand the concept. A class which has not seen enough examples may be perfectly willing to accept $(a \times b) + (a \times c) = a + (b \times c)$ or $a^n + a^m = a^{(n+m)}$.

When you have one or more generalizations on the board, you can demonstrate their power by asking the class to tell you what to erase so that you only keep the statements which tell you the most information or which summarize the day's lesson.

Here are several more examples of generalizations.

(a) The Distributive Law $a \times (b + c) = (a \times b) + (a \times c)$

(b) $k \times \frac{1}{k} = 1$ for $k \neq 0$.

(Note that the question of whether $k \times \frac{1}{k} = 1$ for all numbers k leads to an interesting discussion about multiplication by 0.)

(c) $\alpha \times 1 = \alpha$

(d) $a^n \times a^m = a^{(n+m)}$

It is important for students to realize that not all statements and patterns lead to generalizations. This can be shown by examples such as:

(a) $2^3 + 2^3 = 2^4$ but $3^3 + 3^3 \neq 3^4$

(b) $2^4 = 4^2$ but $3^4 \neq 4^3$.

Another useful activity is for students to make conjectures about possible generalizations and attempt to verify or disprove them. For example, a class which has studied the distributive law $\alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$ might conjecture that $\alpha + (\beta \times \gamma) = (\alpha + \beta) \times (\alpha + \gamma)$.

An excellent exercise is to present several true statements and ask which ones can be generalized and how.

Example: $5 + 0 = 5$ $\quad 2 + 3 = 3 + 2 \quad 2^4 = 4^2$

$5 + 3 = 8 \quad (2 \times 3) + 2 = 2 \times 4 \quad \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

Equivalent Sentences

Replacing a mathematical expression by an equivalent one is often an invaluable aid to solving a problem, understanding a new concept, or arriving at a generalization.

$\frac{1}{2} + \frac{1}{4} = ?$ becomes $\frac{2}{4} + \frac{1}{4} = ?$

$^{-}2 + 5 = ?$ becomes $^{-}2 + 2 + 3 = ?$

$2^3 \times 2^4 = ?$ becomes $(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2) = ?$

$(2 \times 3) + (2 \times 4) = 2 \times 7$ is easier to generalize in the form $(2 \times 3) + (2 \times 4) = 2 \times (3 + 4)$.

The following process helps the students discover the concept of equivalent mathematical sentences in a comfortable situation:

Begin with an open sentence at an appropriate level for the class, like

$$18 + \square = 32.$$

Quickly give the class additional problems like

$(6 + 12) + \square = 32$ $\qquad$ $(9 \times 2) + \square = 32$

$(3 \times 6) + \square = 32$ $\qquad$ $(6 + 6 + 6) + \square = 32$

$(36 \div 2) + \square = 32$ $\qquad$ $(\frac{1}{2} \times 36) + \square = 32$, etc.

Tease the students about being stuck on "14" since they keep giving the same number to different problems. The class will quickly tell you that the sentences are all the same. Introduce the terminology "another name for 18," "another way of writing 18," and "an equivalent expression for 18." Reinforce the concept by having the students make up additional equivalent sentences or repeat the process with another problem.

We have found vertical arrows to be useful to indicate equivalent expressions.

Examples:

$$\begin{array}{ccccc} \frac{1}{2} & + & \frac{1}{4} & = & \square \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \frac{2}{4} & + & \frac{1}{4} & = & \frac{3}{4} \end{array}$$

$$\begin{array}{ccccccc} (2 \times 2 \times 2) & \times & (2 \times 2 \times 2 \times 2) & = & 2^7 \\ \downarrow & \downarrow & \downarrow & \swarrow & \downarrow \\ 2^3 & \times & 2^4 & = & \square \end{array}$$

$$\begin{array}{ccccc} {}^{-}5 & + & 8 & = & \square \\ \downarrow & & \downarrow & & \searrow \\ {}^{-}5 & + & \overbrace{5+3} & = & \square \end{array}$$

Don't allow arrows to outlive their usefulness and become a fetish to the point where a problem is "wrong" in the students' minds if it doesn't have all the arrows. Arrows for operational and relational symbols can be omitted, although they are helpful, at first, in focusing on the renaming process.

The many steps involved in writing an equivalent sentence provide opportunities to involve students with a wide range of skills in the solution of the same problem. Questions such as "How can I rewrite the 8 using a 5?", "What do I bring down for +?", "Where did this $\frac{2}{4}$ come from?", etc., often involve the least active participants in the class.

Frequently an open sentence is equivalent to (asks the same question as, checks, has the same truth set as) another sentence which is easier to solve, e.g.:

(a) $2^5 \times 2^y = 2^0 \Longleftrightarrow 5 + y = 0$

(b) $9L3 = \square \Longleftrightarrow 3E\square = 9$

(c) $10 - 6 = \triangle \Longleftrightarrow 6 + \triangle = 10$

$5 - {}^{-}2 = \nabla \Longleftrightarrow {}^{-}2 + \nabla = 5$

(d) $4 \div \frac{1}{2} = ⬡ \Longleftrightarrow \frac{1}{2} \times ⬡ = 4$

<u>Use of Generalizations to Extend Definitions</u> (Use it in a sentence)

Frequently in mathematics, one is confronted with the problem of defining operations on new sets of numbers or extending definitions to new numbers.

Examples: (a) $\frac{1}{2} \times \frac{1}{3} = ?$

(b) What's a sensible definition of 2^0?

(c) ${}^{-}2 \times {}^{-}3 = ?$

A common approach in discovery classes is to make up a "true" sentence containing the unknown expression and then see what it "acts like." For the above cases, the sentences might be

(a) $$\frac{1}{2} \times \frac{1}{3} \times 3 \times 2 = 1$$
$$\square \times 6 = 1$$

(b) $$2^0 \times 2^3 = 2^{(0+3)}$$
$$\square \times 8 = 8$$

(c) $$(^{-}2 \times {}^{-}3) + (^{-}2 \times 3) = {}^{-}2 \times (^{-}3 + 3)$$
$$\square + {}^{-}6 = 0$$

Note that each sentence is true because we assume a property derived for positive integers is also valid for the integers and rationals. In the first example, we assume the commutative and associative properties for multiplication; in the second, the formula for multiplying exponential terms; and in the third, the distributive law. These assumptions generally will be tacit at first, although through language such as "If this sentence is true, what does that tell us about $^{-}2 \times {}^{-}3$", the class can gradually appreciate the process by which mathematicians extend definitions to larger number systems.

Patterns

Patterns appear frequently in mathematics. They often suggest a conclusion or make results derived through some other method seem more plausible. Patterns are helpful for reinforcing and reviewing previously learned concepts. Since patterns can be misleading, care should be taken to verify results obtained via patterns with another approach and to introduce the students to some misleading patterns.

Examples:

(a) Reinforce 2E0 = 1 by looking at:

$$\begin{array}{lll} 2E3 = 8 & & \\ & \div\ 2 & \\ 2E2 = 4 & & \\ & \div\ 2 & \\ 2E1 = 2 & & \\ & \div\ 2 & \\ 2E0 = ? & & \end{array} \qquad\qquad \begin{array}{ll} 2E3 = 8 & \\ 2E2 = 4 & \times\ 2 \\ 2E1 = 2 & \times\ 2 \\ 2E0 = ? & \times\ 2 \end{array}$$

(b) Derive multiplication of negative numbers using the distributive law. Reinforce via

$$\begin{array}{rcrcr} {}^{-}3 & \times & 4 & = & {}^{-}12 \\ {}^{-}3 & \times & 3 & = & {}^{-}9 \\ {}^{-}3 & \times & 2 & = & {}^{-}6 \\ {}^{-}3 & \times & 1 & = & {}^{-}3 \\ {}^{-}3 & \times & 0 & = & 0 \\ {}^{-}3 & \times & {}^{-}1 & = & 3 \\ {}^{-}3 & \times & {}^{-}2 & = & 6 \end{array}$$

(c) Use the additive law for exponents to establish $2E^{-}1 = \frac{1}{2}$ and $2E^{-}2 = \frac{1}{4}$. Extend to $2E^{-}3$ etc., via

$$\begin{array}{lcl} 2E4 & = & 16 \\ 2E3 & = & 8 \\ 2E2 & = & 4 \\ 2E1 & = & 2 \\ 2E0 & = & 1 \\ 2E^{-}1 & = & \frac{1}{2} \\ 2E^{-}2 & = & \frac{1}{4} \\ 2E^{-}3 & = & \\ 2E^{-}4 & = & \end{array}$$

(d) Some misleading patterns:

(i) $2^2 = 2 \times 2$

$2^1 = 2 \times 1$

$2^0 \stackrel{?}{=} 2 \times 0$

(ii) 3 is prime, 5 is prime, 7 is prime, 9 is prime?

(iii) 6E2 = 36

5E2 = 25

4E2 = 16?

(e) A challenging pattern:

$1^2 + 1 + 41 = 43$, prime

$2^2 + 2 + 41 = 47$, prime

$3^2 + 3 + 41 = 53$, prime

$4^2 + 4 + 41 = 61$, prime

$5^2 + 5 + 41 = 71$, prime

Is $n^2 + n + 41$, prime for all n?

<u>Equality; Transitivity</u>

To many students, "=" is a command which says "do the problem" or "find the answer." Thus they may be uncomfortable in accepting such statements as 6=4+2 or 2x2x2=2E3. They may also have difficulty drawing conclusions from a long chain of equalities. This section discusses things the specialist can do to clarify the meaning of "=" to students.

A useful exercise in helping students understand what equals means is to write on the board:

2 + 4 3 x 2 and ask what symbol could be placed between the two expressions to make a true sentence. Once a student suggests equals and the class agrees, have the students think of other expressions besides 3 x 2 that are <u>equal</u> to 2 + 4.

Now, again ask what "=" means. Probably students will suggest: the same as, or, means the same number. This understanding of equality will be useful when you work with equivalent sentences.

Drawing conclusions from a chain of equalities often does not come naturally to students. They are perfectly comfortable writing a chain of equalities like $2 \times 6 = 12 \div 3 = 4 \times 9 = 36 \div 6 = 6 \times 3 = 18$ without recognizing it is logically absurd.

To clarify meaning of a sentence with more than one equal sign in it, you might want to discuss transitivity in general:

(a) If $\square$ has the same number of candy bars as $\triangle$ has, and $\triangle$ has the same number of candy bars as $\bigcirc$ has; does $\square$ have more or fewer candy bars than $\bigcirc$?

(b) If John is the same age as Joe, and Joe is the same age as Stan, what is true about Joe and Stan?

(c) If $\sum_{i=1}^{4} 3 \times i = 3 + 6 + 9 + 12$ and $3 + 6 + 9 + 12 = 30$, what is the numerical value of $\sum_{i=1}^{4} 3xi$? How did you know?

(d) If $\square = \triangle$ and $\bigcirc = \triangle$, then what symbol can I put between $\square$ and $\bigcirc$? Check by filling in the shapes with numbers or expressions.

(e) If a = b and b = c and c = d and d = e, what can you say about a and e?

Once you have done many problems and examples, you might want to develop a standard expression for drawing conclusions, such as "the conclusion of the chain of equalities is . . ." (COTCOE)

The generalization (aEb) Ec = aE (b x c) is an example of a concept that results from considering a chain of equalities. In trying to develop this concept, you might consider: (2E3) E4 = (2E3) x (2E3) x (2E3) x (2E3) = 2E (3+3+3+3) = 2E (3 x 4); you wish the class to draw the conclusion (2E3) E4 = 2E (3 x 4). Some suggestions for leading the class to this result are:

(a) Who can find another name for (2E3) E4 that uses the least chalk?

(b) Who can find a name for (2E3) E4 that uses only a 2, 3, and 4 once as in the problem?

(c) Who can point to something on the board that's equal to (2E3) E4? Keep having students do this until you get the 2E (3 x 4) form.

(d) Enclose the first and last parts of the sentence in a simple closed curve, and have the class read what is in the curve.

(2E3) E4 = (2E3) x (2E3) x (2E3) x (2E3) = 2E(3+3+3+3) = 2E(3x4)

(e) Rearrange the problem with the help of vertical arrows. Have the students read the arrows as equals. Finally ask:

"What symbol is missing?"

(2E3) E4　　　　　　2E (3x4)

(2E3)x(2E3)x(2E3)x(2E3) = 2E (3+3+3+3)

(f) Discuss transitivity in general (see above).

Embedding the Material in a Conceptual Framework

One of the most frequently used strategies in discovery teaching is to embed elementary material in a more advanced, conceptual framework. Students become motivated to understand topics, which seem to have little inherent interest, when these topics are essential to the solution of conceptual mathematical problems. Several examples involving fractions and decimals are described below.

To a mathematician, fractions and decimals are each ways of looking at the rational numbers, but each leads to different insights. It so happens that decimals are really a kind of infinite series, that is, each decimal is so many tenths, plus so many hundredths, plus so many thousandths, plus etc. Most decimals we run into terminate, such as in .25 or .456, but sometimes they do not, as in .2222222222222, with an unending string of 2's, and since decimals such as this last one are adding up an infinite number of terms, doesn't it seem sensible that they should increase without bound? In fact 2.3 is bigger than 2.22222222222 (with an infinite number of twos). Everybody knows this, but how many can explain why? From the point of view of a discovery teacher, this can be fruitful territory.

Let's examine this example more closely. First it is essential that we as teachers fully understand the concept we are trying to get across, and then it is necessary that we trace the process we went through to arrive at our conclusions, so that we can devise a sequence of questions. So why is 2.2222222222222 less than 2.3? Well, 2.3 = 2.30. Why? Because $2\,\frac{3}{10} = 2\,\frac{30}{100}$. We can carry this further, however, and see that 2.3 = 2.300000000000, with as many zeros as we want, and now it is easier to see why 2.3000000000000 is greater than 2.2222222222222, and hence why 2.3 is greater than 2.22222222222222. But that still leaves us with the fact that you can add up a series like $2 + \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} +$ etc., and that you will be bounded from above by something as small as 2.3.

A simpler but related problem begins by looking at what happens when you add $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$: Do you reach any sort of a limit? Have the students try some examples to find out. They will see a pattern, and you will arrive at the conclusion that you get closer and closer to 1. Now try it for $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ You will find that the answer gets closer and closer to $\frac{1}{2}$. Now see if the students can see a "pattern in the patterns," that is, for a given series, can they predict the answer.

Discussions such as these will help students to see why they can add zeros behind the decimal point and will provide practice in adding fractions, and this is, after all, one of your practical goals. The point here is that, in the course of their discussions about conceptual matters, students will be continually calling upon the basic arithmetic facts you want them to learn. In this way, they begin to master the facts, but they tend not to experience the drudgery.

Another problem which motivates the study of fractions is the "infinite pie" idea. What happens if you keep cutting a pie in half? Do you every run out of pie, even if you keep cutting forever? You might come to a point in your discussion when students are talking about atoms, and they may realize that they need to consult an encyclopedia after class. This is physics, not math, but they are motivated, and they will know they have been discussing something that is not trivial.

CHAPTER 4

FEEDBACK AND INVOLVEMENT

This chapter contains a number of techniques for getting feedback from a class and for increasing the level of student participation in the lesson. Both are essential to effective discovery teaching. In order to ask appropriate questions, you need to know the class level of understanding. In order to gain from the discovery process and give you feedback, students need to be involved in the lesson. Frequently, the same technique accomplishes both feedback and involvement, hence we have included both in this chapter.

Circulating

Circulating the class is an important way of getting feedback from the students. It also is useful for involving students and giving the teacher a chance to make individual contact with them.

At least four times in the class period, the students should be asked to answer a question or work a problem on their papers. Be sure that each student writes something, even if it is only a question mark. Allow the students to signal you by raising their hands when they are finished. Circulate rapidly, making some contact with each student. In general, do not comment on whether the answer is right or wrong, or provide individual tutoring. (You do not want to single out students with the wrong answer, or spend so much time you lose the class's attention.)

It is not necessary to read every answer each time. You might check only two rows and then make a prediction about the percentage of the class that will have the answer correct.

Circulating gives you accurate feedback on student understanding, helping you gauge when to move on to a more advanced topic. If most of the students were correct, then the answer can be put on the board and the class can move on. If not, you might want to have the class investigate the different ideas about the problem by offering all answers, correct or incorrect.

Circulating makes all of the students active participants, often exposing misunderstandings you and the class were unaware of. It also gives you an opportunity to involve students who understand the concepts but are unsure of themselves or are reluctant to respond. Such students can be reinforced by asking them to put their "absolutely correct" answer on the chalkboard. One way to be certain to remember is to leave the chalk with them when you circulate.

Circulation as it is described here is a technique for gaining accurate feedback from the entire class without disrupting the flow of the discovery lesson. Questions should be relatively straightforward ones that can be answered with a simple number or phrase, or should be problems that can be solved quickly. More complex problems or sequences of practice problems should be saved for a "seatwork" period during which you and students who have completed their work might offer assistance, or students might work together in small groups.

Polling

When a student gives an answer, quickly ask for a show of hands--"How many of you would have said that?", or "How many agree with Jane?" Although the feedback will not be as accurate as from circulating, as some students will go along with the majority, this is rarely a serious problem. You can get a good sense of how many students actually have the answer by how rapidly the hands go up. Because you get the response quickly, there is time to ask similar questions. Often a student who merely copies his neighbor begins to focus on the problem and soon begins answering questions spontaneously.

Another variation of the polling technique is to put several answers on the board and have the students vote on them.

You might also rapidly get answers from a number of students. Again, students who are unsure of themselves may jump on the backwagon and gain confidence as they are recognized for having the right answer.

Finger and Hand Signals

Finger and hand signals enable the teacher to get a rapid reading of the class, and they allow many students to participate at once.

Do not be overly concerned about the problem of copying when answers are shown in fingers. You can get a good sense of how well the class is understanding by noticing how quickly and confidently they respond. If truly accurate feedback is desired, the students can be asked to close their eyes or write their answers on paper. On other occasions, when fingers are used to obtain mass participation, slower students can participate along with the rest of the class. They are more likely to focus on the problem and catch up to the rest of the class. Be careful about putting a student on the spot by asking him to explain an answer he has on his fingers. Volunteering to show an answer is quite different from volunteering to give an explanation.

Examples of signals often used follow:

(a) Numerical values can be shown on the fingers.

(b) Other mathematical symbols such as operational or relational symbols can be shown on the fingers or drawn in the air.

Note: Younger students genuinely enjoy showing answers with their hands. Older students tend to think of it as childish, so it should be used sparingly. It can, however, be inserted occasionally in a lighter moment or as a change of pace.

(c) Signals for agreement and disagreement should be developed. Younger students like waving their arms back and forth for disagreement and pumping both fists for agreement. With older students, head nods and shakes are often the best signals. You can ask the students to indicate whether they agree or disagree every time a question is answered.

Chorus Response

Often, during a lesson, ask the students to tell you their answer together, or to read in chorus something from the board. This creates a break in the normally quiet question-answer pattern. It focuses attention and brings back daydreamers. Reading provides an opportunity for students who don't yet understand the mathematics to participate. Chorus response allows the entire class to move rapidly through a sequence of questions leading to a conclusion.

In order to avoid chaos and called out answers when you don't want them, it is helpful to cue the class when you want a whole group response. "Class, what is . . .," "When I count to three, everyone tell me . . .," and "When I drop my hand, everyone with his hand up tell me . . ." are ways of achieving this. When you want to return to a different mode of response, begin your question with a direction such as "Raise your hand if you know . . ." or "Write on your papers . . . ."

Some situations in which chorus responses are appropriate and effective are:

(a) Emphasizing a concept or generalization that has just become clear.

(b) Focusing on a problem that has just been put up on the board.

(c) Learning new words or symbols.

(d) In response to short, fast-paced questions, such as simple computations, gradual escalation to a generalization, or review questions.

(e) To refocus the class after an interruption.

(f) To involve nonparticipants.

(g) To summarize the day's work. At the end of the class, have students read the most important words and concepts on the board, while the class makes sure their notes are in order.

Rapid Oral Questions

A round of rapid-fire, oral response questions provides a change of pace, while many students get a turn to respond in a short period of time. The questions should be easy, or follow a pattern, so everyone will feel capable of answering. Some of the best times to use a rapid series of oral questions are when reviewing, when building up to a generalization, or when doing a routine calculation as part of a larger problem.

Examples: What's $\frac{1}{2}$ of 8? $\frac{1}{2}$ of 16? $\frac{1}{2}$ of 32? etc.

In 9,876,543.21, what place is the 4 in? The 5? The 8? The 2? etc.

Counting, Naming, and Predicting Hands

Counting the number of hands raised or mentioning the names of students with raised hands often increases the number of students who participate. An auctioneering style: "Ten people have it," "This whole row," etc., or naming students: "John has it," Sue's got an idea," etc. are both effective for this purpose. Students want the recognition you are giving, and the time you spend talking about hands gives them extra time to think.

To avoid putting an insecure student on the spot, it's best to call on one of the first hands or the entire class for the actual answer. The same student, however, might be called on to second another student's answer. Once they are focused on the problem and participating in some way, reluctant students frequently are eager to answer the next question.

A variation of these techniques is for you or one of the students to predict the number of students who will be able to answer the question. You might challenge them: "I bet only ten people will be able to solve this."

Counting and predicting provide an excellent opportunity to review percents: "What percent of this row has their hands up?"; "If 75% of the class gets the problem, how many students will that be?"

Chain Answering

Chain answering is an effective way to involve as many students in answering a question as there are steps in the problem. "Cheryl, would you start us off?" "That's good, I know you can do the rest. Call on someone with a raised hand to do some more." Each time you stop a student, she quickly calls on another student.

A variation has students come to the board and hand the chalk to the next student when their turn is finished.

Examples of situations where chain answers are effective are:

(a) Reading a complex generalization such as

$$(\alpha \times \beta) + (\alpha \times \gamma) = \alpha \times (\beta + \gamma)$$

(b) Substituting numerical values in a mathematical formula such as $a^n \times a^m = a^{(n+m)}$

(c) Building a table such as powers of 2.

(d) Providing the next term or problem in a sequence such as $\frac{1}{3} + \frac{1}{6} = ?$, $\frac{1}{3} + \frac{1}{9} = ?$, $\frac{1}{3} + \frac{1}{12} = ?$ etc.

(e) Working each step of a multi-step problem such as

$$5 \times \frac{1}{3} \times 3 \times 2 \times \frac{1}{2} \times \frac{1}{5} \times \pi = ?$$

$$1 \times 1 \times 1 \times \pi = ?$$

$$1 \times \pi = ?$$

$$\pi = ?$$

Deliberate Errors

Successful discovery teachers make deliberate errors at judicious points in the lesson. Deliberate errors serve to increase student participation, to reinforce mathematical understanding, and to build student confidence. They force students to evaluate each response critically and not to rely on the teacher or other students. Students delight in catching the teacher in a mistake. At the same time, the teacher's error removes the stigma of being wrong, and thus makes students more willing to venture an answer. Explaining their disagreement helps students to demonstrate and articulate their understanding.

The simplest errors are clerical, such as transposing two numbers, miscopying a problem, or misspelling a word. They serve primarily to focus the class and keep them alert. Sparing use of this technique, coupled with a warning that there may be some mistakes in your boardwork, avoids the danger of seeming patronizing.

A second category of errors involves literal interpretation of student answers.

Example: $2^3 = 2 \times 2 \times 2$

$2^5 = ?$ Student response: "You add two more 2's."

You write $2^5 = 2 \times 2 \times 2 + 2 + 2$.

When the students are secure in their understanding of a concept, errors of this type help them to communicate their knowledge better.

The final category of errors are conceptual ones, often focusing on the same types of errors the students tend to make. These errors generate enthusiasm, debate, and discussion, but most important of all, reinforce understanding.

Examples:

<u>False Statement</u>	<u>Devious Defense</u>
$2^3 \times 2^4 = 2^{12}$	The "x" tells you to multiply.
$2^0 = 0$	2E3 = 2 x 2 x 2, but 2E0 = . And nothing is zero.
$2\text{E}^{-}1 = 1$	$2 + {}^{-}1 = 1$ (The best arguments for false statements are often those originally presented by the students in their innocence.)
$5 + {}^{-}5 = 5$	Negative numbers do not exist.
$\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$	1 + 1 = 2 and 2 + 4 = 6. And that has to be the way to work the problem, because how do you work $\frac{1}{2} \times \frac{1}{4} = \boxed{}$?

Attention and Focus

Student attention is bound to wander. This happens for many reasons: noise distractions, intercom interruptions, people going in or out, earlier incidents. Besides these non-SEED reasons, students who are experiencing some difficulty with the mathematics will tend to "drop out" from class participation. A student who hasn't heard or wasn't listening for whatever reason will not be raising his hand. It is OFTEN necessary first to catch student attention, before you pose your question.

Each technique described below is useful for either the whole class or for individuals. Variety is essential. Any participation techniques, especially chorus responding, also help students focus and pay attention since they involve the students in a response. Other techniques whet the students' appetite and motivate their interest through elements of suspense, challenge and contradictions.

1. ARE YOU READY? CAN YOU SEE THIS?

Before you ask the math question, ask the class or certain students: "Are you ready?"; "Who is ready for a hard problem?"; "Who thinks they will be able to do the next problem?"; "Do you think you are ready to make a quantum leap to the generalization?"; "Do I have everybody's attention--no, someone is looking out the window."

Specify some mode of response to your question: a specific way for them to show you they are. "Raise your hand; say 'Yes,' if you are ready." If not many students respond, repeat your instruction till you get a stronger (faster, more unanimous) response.

2. <u>VOICE EMPHASIS</u>

<u>Any</u> CHANGE in your voice will gather attention. When it is especially desirable to emphasize an important part of a student's idea or of the question, either suddenly or gradually talk more rapidly or talk very de-lib-er-ate-ly; speak much louder or softer; speak with a higher or lower pitch. Stating your question with pauses stimulates anticipation, attention and excited responses: "What is . . . the numerical value . . . of 4E2?"

3. <u>CHALLENGE</u>

Challenging is a form of teasing. Properly timed and with the proper tone, a challenge will energize the students and increase the focus they give to the upcoming problem. Choose problems you believe students can do. With too hard a problem or the wrong tone, students may be discouraged from trying the problem.

Comment before you give the problem: "This one may be too hard."; "I'll catch you on <u>this</u> one."; "This is tricky, only the most alert will get it."; "Last time I gave this problem, no one got it."; "You need to switch into a higher mental gear."; "In fact this is a star problem."; "This is a no hand problem."

When students are secure about their answer, you might challenge them by claiming the answer is wrong and forcing them to defend it.

Obvious contradictions and "double talk" are also effective for revitalizing a class. Tell the students to forget something, then ask them to tell you what they were supposed to forget. "Don't anybody listen to what Cheryl just said because she is absolutely right!" "This is so important, I'm going to erase this before anyone has a chance to write it down."

4. <u>CUED RESPONSES</u>

Occasionally ask the class to do something on a certain predetermined signal. If you stall before giving the cue, the time that passes before the cue or signal is given builds up the class's anticipation towards making the agreed upon response: almost invariably you will have nearly 100% participation at the cue. Some examples of cues include: "When I turn around . . .;" "When I lower the chalk (or the eraser) from over my head;" "When the chalk touches the blackboard;" "When I count to three;" "When I close the door;" "When Tanya raises her hand." You can use this as a daily technique so long as the cue is a different signal each time you use it. Younger students enjoy your trying to catch them on a false cue, such as touching the board with a finger instead of the chalk, counting "1, 2, 4," skipping over "3," or lowering the hand without the chalk.

With younger students, a very powerful technique to refocus their attention after a major disruption has occurred is a series of fast directions which appear to have no point until the end, when you are pointing to the problem that you wish to ask a question about. It goes like this: "Look out the window, OK, now look at the door, now look at the clock, read this word on the board, read this sentence, read this problem."

5. <u>POINT TO AN ANSWER</u>

This involves every student, focuses their attention on some particular mathematics, and usually wakes up any day-dreamers, who can then pick up from their classmates what is going on. Ask the students to point, from their seats, to: a sentence on the board that we could use in solving this problem; a place where 3 goes in this sentence; the chart that shows the answer. Tease a student by going back to a student's desk to take a sighting along his/her finger, then follow it to some spot on the board.

6. BOX OR CIRCLE

Box, circle or draw an arrow ---> to make the problem or question currently under investigation stand out. The visual reference will help a non-attender pick up where the class is. Box all important problems which emphasize the conclusions the class arrived at in that lesson. Let students come to the board to circle a hint, or circle what told them to use "X" for the factors.

7. READ

Write a problem, ask someone to read it while everyone else is thinking about it. Sometimes ask the whole class to read a problem as you write it on the board.

To get a class to scan the blackboard for helpful information, make an elaborate show of erasing any clues or hints on the board, telling the students you don't want to give away the answer. You intend, of course, to help them focus on a clue. You might erase something close to the clue or erase the hint itself. Vary this by circling clues, covering them up, saying "Don't look for any hints" (they will then study the board, whether there are any hints or not). On a very difficult problem, you might want to reverse this technique by erasing everything except the clue: "I'm going to erase all of the irrelevant information."

CHAPTER 5

BUILDING STUDENT CONFIDENCE

A major goal of discovery teaching is to build students' confidence in their ability to think critically and to learn mathematics successfully. This is particularly true in general mathematics classes, where the students often have nine or ten years of school experience convincing them that the opposite is true. You want them to be involved in the lessons, building the mathematics themselves. You want them to experience success and to regard what they have accomplished as having value.

An instructor who applies the principles of the preceding chapters is well on the way to achieving this goal. There will be numerous conceptual mathematics questions which students can answer successfully. Frequent review will help the students maintain mastery of an ever-increasing body of mathematics. Feedback and involvement techniques will provide ample opportunities for the class as a whole to participate.

The manner in which you treat student responses will influence their willingness to answer future questions. This chapter presents ways to respond to student answers, and techniques that will establish an atmosphere of intellectual support and respect in which students feel free to participate and are rewarded for it. Coupled with the question and feedback techniques discussed previously, they create a classroom in which students learn mathematics with a sense of accomplishment and achievement.

Success Reinforcement. Generally, if a student responds with a correct answer, he or she will be rewarded by you or the class. It's hard not to show your pleasure (although it often heightens the student's experience of success if you act noncommittal and ask the class its opinion). Some ways of reinforcing and creating success experiences are outlined below:

1) Use names. Identify ideas and formulas by the names of students who developed them. "Who could do this problem using John's system?"; "Let's use Sue's method for finding the LCM," etc.

2) Student agreement. After a response from a student, ask the class if they agree. A roomful of raised hands is definitely reinforcing.

3) Acknowledge other students. Mention the names of students who have their hands raised when someone else is called on to answer, or who are showing agreement or disagreement. "Mary has the answer;" "John has it too." "Joe agrees with Jane;" "How many got 3/4?" etc. Eye contact or a smile can also let a student know that you know that she knows the answer.

4) Students to the board. Ask students to show their work on the board. This technique is especially reinforcing when you call on a normally shy or non-participating student whose correct answer you have just seen while circulating. If several students come to the board at the same time or in rapid succession, have them put their names by their work.

5) I know you know. When you are sure a student understands a concept, interrupt his explanation with "Good, I know you understand this." Then invite him to call on another student to finish the explanation. A variation allows you to bypass a student who is threatening to monopolize the class. "I know you know the answer, Jane; I want to see what Jerry thinks;" or "John, call on someone else you think knows the answer."

6) Experts. Designate students, who have caught on to a concept, as experts for the day. Have them check the correctness of other students' answers, or help you circulate. These students can also be involved as peer teachers or tutors.

7) Star problems. Label occasional challenging questions which are within the grasp of the class as "star problems." Put a star on the board, and next to it put the names of students who solve the problem.

Many instructors also give similar recognition for students who ask good questions.

8) Advanced material. Praising students lavishly for trivial work is patronizing and they recognize it. If you embed remedial work in advanced material, you can give students a real ego boost.

Examples:

$$\begin{array}{ccccc} 2^{-1} & \times & 2^{-2} & = & 2^{-3} \\ \downarrow & & \downarrow & & \downarrow \\ \frac{1}{2} & \times & \frac{1}{4} & = & \frac{1}{8} \end{array}$$

"Normally students don't study this until algebra."

$$\sum_{i=1}^{3} 2^{-i} = \begin{array}{ccccccc} \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & = & \frac{7}{8} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \frac{4}{8} & + & \frac{2}{8} & + & \frac{1}{8} & = & \frac{7}{8} \end{array}$$

"I didn't study this until I was in college."

9) Who has it now? Give students a chance to indicate their progress in learning an idea, particularly after they have met with initial frustration. Ask: "Who has the answer now ("Who sees how to do this?") Who didn't have it before?"

10) <u>Reward questions</u>. Mathematical research proceeds when mathematicians ask themselves questions. A student in a discovery class who asks a good mathematical question should be praised. Students should be encouraged to ask themselves if alternate approaches can be used for solving a problem, or what happens if the problem is changed. Occasionally, the volume of good questions threatens to slow down the pace of the class because too much time is spent investigating tangents. In this case, ask the students to write their questions on paper and hand them in. The few minutes that it takes to write a response can be richly rewarded in the ensuing mathematical correspondence that will develop.

<u>Student errors</u>. The manner in which a student's incorrect answers are handled will influence his willingness to respond to future questions. Try to avoid telling a student directly that he's wrong, or putting him on the spot.

Some suggestions for handling incorrect answers follow:

1) <u>Deliberate errors</u>. If you make frequent deliberate errors, there will be no stigma attached to being wrong. Students will disagree with you or each other matter-of-factly.

2) <u>Allow revisions or call on someone</u>. Put incorrect answers on the board with a straight face. Let the class disagree politely. Allow the student to revise her answer or to call on another student who she thinks knows the answer. After you've gotten the correct answer, return to the student and ask if she understands now.

3) <u>Explore consequences</u>. Often technically incorrect answers contain good thinking and mathematical creativity. Students often find that by slightly changing the question or one of the assumptions (axioms) of the problem, they can more easily answer the question. Research mathematicians are rewarded for this kind of thinking. Students are usually told they are wrong and made to feel foolish.

Good discovery teachers turn technically incorrect answers into a positive learning experience. If the student has changed the question, such as $\frac{1}{2} + \frac{1}{4} = \frac{1}{8}$, ask "What problem did Sue solve? or "What was Sue thinking of?" Sue gets credit for having solved a problem correctly and the class benefits from focusing on the contrast between Sue's problem and the original problem. If the student has changed the axiom system, say 2E3 = 23, you might give the system his name, and ask the class to do some more problems in "Carl's system." In both cases, the student's response has provoked a fruitful class discussion, and he will be more willing to offer an idea in the future.

If you can see the reasoning behind the student's answer, you might want to present the argument to the class and let them disagree with you, rather than the student. For example, if a student tells you 2x2x2=6, you might argue that "2 times 2 is 4 <u>plus</u> 2 more is 6."

4) <u>Partial answers</u>. If a student has worked some steps of a multistep problem correctly, be sure he gets credit for it. Focus on the correct parts: "How many agree with this step?" and have the class redo only the part that needs correction.

Encouraging insecure students. In addition to the group techniques for encouraging student participation, there are a number of techniques for building confidence in shy or insecure students. These are particularly helpful at the crucial moment when a student you have called on starts to falter in a response or explanation. There are several techniques for helping him out of a potentially embarrassing spot without his losing face. Some suggestions follow:

1) Techniques for handling errors. (See above section.)

2) Hints. Often a student who is hesitating needs only a hint to continue the answer. Have him or her call on someone to give a hint.

3) Rephrase the question. Restate the question so that the answer is more apparent, or ask a subquestion, or ask for a student volunteer to do so. For example,

   Question: "What's the factor form of 2E3?"
   Restatement: "How many times do I use 2 as a factor in 2E3?"

   Question: $\frac{1}{4} + \frac{3}{8} = ?$
   Subquestion: "Can you shade in $\frac{1}{4}$ of this circle?"

4) Call on someone to work with you. Have the student call on another student to work with him. This is particularly effective when a student gets stuck at the board.

5) Can you tell anything about it? Encourage the student to make some contribution or estimate toward solving the problem. "Can you tell me something about part of it?" "Is it bigger than 10?" "Is it negative?"

6) <u>Call on someone</u>. If the student still doesn't want to try the problem, have her call on someone else.

7) <u>Come back with success</u>. Try to come back to the student during the same class period with a positive experience. Call on him for a question he can answer successfully. Acknowledge his hand when it is raised. Give him a special responsibility, like recording an important result for the next day. Use any of the techniques for involving non-participants listed elsewhere.

8) <u>Preteach</u>. Work with insecure students for a few minutes outside of class. Teach them something the class doesn't know, such as a new Greek letter or the next step in a problem the class is working on. When the opportunity arises in class, they have a chance to star.

9) <u>Encourage questions</u>. Frequently praise students who ask questions when they don't understand something. Point out that asking for help when you don't understand is an important step in learning. Often if the student basically understands, asking him another question will enable him to clear up his own confusion. Asking how many other students have the same question often relieves the pressure of not knowing.

<u>Student interactions</u>.

1) <u>Listening to each other</u>. It is important to make certain that students listen to each other and do not rely on you to determine the correctness of an answer. The signals for agreement and disagreement discussed earlier facilitate this. You can also encourage it

by insisting that students speak so that they can be heard. Do not repeat inaudible answers, but say something like "Joe, did you hear Jane's answer?"; "Sue, repeat your answer so John can hear it."; or "Who can repeat what Bill said?"

2) Mathematical debates. Many people recall taking classes in the humanities in which they were encouraged to debate and discuss ideas with their fellow students, but they do not realize that it is possible to have the same experience in a math or science class, because in mathematics and the sciences "everything is so exact." This is emphatically not the case, however, for in mathematics and the sciences there is much that is ambiguous. The classic example in the Project SEED curriculum concerns the notion that any number raised to the zeroth power is one. It turns out that this is a highly ambiguous "fact," and that it can provide much fruitful debate. Just as in a discussion of philosophy or theology, students must carefully weigh the merits of various points of law. They must call upon their powers of persuasion and reasoning to sway other students towards their point of view. In a good debate everyone is participating in some way, from the vocal students who are doing much of the talking, to the quiet students who are being exhorted to take sides.

A good discovery teacher must be adept at keeping a debate lively and focused. Sometimes it is necessary for the teacher to introduce material into the discussion that will either clear up intellectual logjams, or rattle the foundations of student arguments. All the while, the teacher is trying to orchestrate the discussion so that students will arrive at a lucid understanding of the concepts.

Once teachers who use the discovery style have seen the benefits to be derived from debates, they will be on the lookout for likely topics. "Mini-debates" are possible on just about any subject, such as which way to move the decimal point when converting decimals into percents, or whether you need a common denominator when you multiply fractions, or whether there is such a thing as a square root that is not irrational, but not a whole number either. By being constantly called upon to debate and discuss, students will begin to realize that mathematics is not an obstacle course filled with facts that must be memorized by rote, but rather a subject that is fascinating in its form and structure.

APPENDIX

Checklist of Discovery Techniques

I. General Socratic Strategies

A. Effective Questioning Techniques.

1. Write out question sequence.
2. Keep a log
3. Vary difficulty
4. Vary pace
5. Vary response
6. Many questions on same concept
7. Patterns
8. Parallel problems
9. Be specific
10. Open questions
11. Don't push results
12. Students answer students
13. Don't disguise statements

B. Review.

1. Rapid summary
2. New directions
3. Provocative questions
4. Foundation for new material
5. Embed in new material
6. New notation or terminology
7. Student questions
8. Deliberate errors
9. Mid-lesson reviews
10. Plan at end of class
11. Worksheets
12. Change of pace
13. Involvement techniques

C. Vocabulary.

   1. Introduce in context
   2. Write words on board
   3. Vocabulary page

D. Boardwork.

   1. Organize space
   2. Erase all but important results
   3. Circle for focus

II. Mathematics Development Structures

   A. Gradual Escalation to a Generalization
   B. Equivalent Sentences - Vertical arrows
   C. Use generalizations to extend definitions (Use it in a sentence)
   D. Patterns
   E. Equality, transitivity
   F. Embed in conceptual framework

III. Feedback and Involvement

   A. Circulating
   B. Polling
   C. Finger and Hand Signals
   D. Chorus Response
   E. Rapid Oral Questions
   F. Counting, Naming, Predicting Hands
   G. Chain Answering
   H. Deliberate Errors
   I. Attention and Focus
      1. Are you ready?
      2. Change voice
      3. Challenge
      4. Cued responses

5. Point to an answer
6. Box or circle
7. Read

IV. <u>Building Student Confidence</u>

A. <u>Success Reinforcement</u>.

1. Use names
2. Student agreement
3. Acknowledging other students
4. Students to the board
5. I know you know
6. Experts
7. Star problems
8. Advanced material
9. Who has it now?
10. Reward questions

B. <u>Student Errors</u>.

1. Deliberate errors
2. Allow revision/Call on someone
3. Explore consequences
4. Partial answers

C. <u>Encouraging Insecure Students</u>.

1. Handle errors positively
2. Hints
3. Rephrase questions
4. Call on someone to work with you
5. Can you tell anything?
6. Call on someone
7. Come back with success
8. Preteach
9. Encourage questions

D. <u>Student Interactions</u>.

1. Listening to each other
2. Mathematical Debates