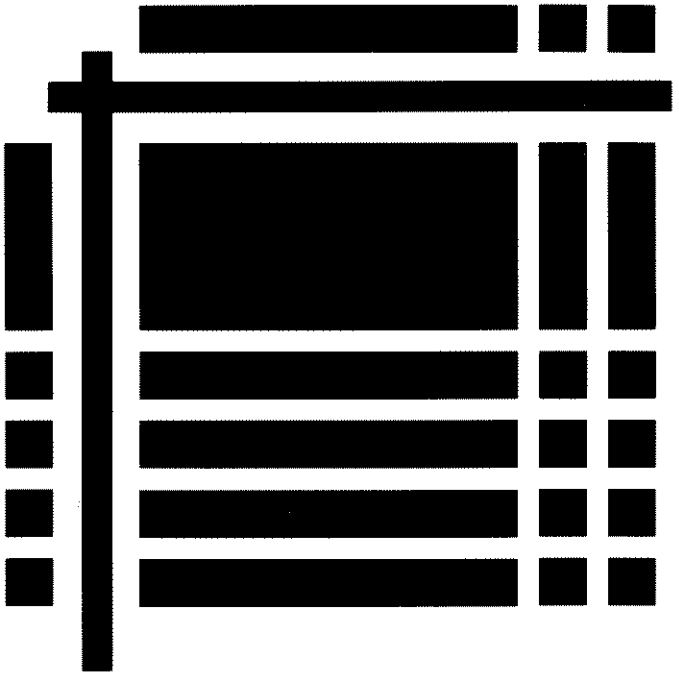
THE ALGEBRA LAB™

High School

A COMPREHENSIVE MANIPULATIVE PROGRAM FOR ALGEBRA I



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THE ALGEBRA LAB™: High School

This book is dedicated to Mary Laycock, one of the great teachers of our time, who has done so much to promote the concrete approach to the learning of mathematics.

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Edited by Linda Charles

Acknowledgements

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The Algebra Lab™

The Algebra Lab™ is a complete manipulative program for teaching algebra concepts. It combines a unique algebra manipulative, The Lab Gear™, with two 144-page activity binders. The first binder, *The Algebra Lab™: Exploring Algebra Concepts with Manipulatives*, is intended for middle school students. The second binder, *The Algebra Lab™: A Comprehensive Manipulative Program for Algebra I* is intended for high school students. Either binder, combined with the Lab Gear, provides a powerful program for introducing, building, and extending the major topics of algebra.

The Lab Gear is an exciting manipulative designed to model algebra concepts. The set consists of ten types of blocks—three constant blocks (1-blocks, 5-blocks, and 25-blocks), and seven variable blocks (x-blocks, y-blocks, $5x$-blocks, $5y$-blocks, x^2-blocks, y^2-blocks, and xy-blocks). There is also a corner piece that helps students organize multiplication and division problems into rectangular arrays. The set of blocks comes in a specially-organized tub with enough blocks for three to six students.

There is a companion set of Algebra Lab Gear for the Overhead Projector. This transparent set of blocks allows the teacher or student to demonstrate examples, solutions, or explorations on the overhead projector. A unique classroom technique utilizes two projectors—the teacher (or student) at one projector, a student at the other. The teacher manipulates the blocks, while the student records with algebra notation, or vice versa. A great concrete-abstract bridge!

All blocks and activity binders are available from Creative Publications. Please see a current catalog for prices and ordering information.

The Philosophy

Algebra is the gate into high school mathematics and science classes, which in turn, are required for college admission. Few courses play as important a role in determining the future options of our students. And yet, many students never get the opportunity to take algebra, and among those who do, there is a high percentage of Ds and Fs. The Algebra Lab is an attempt to help change this grim situation. By offering a concrete, informal preparation for later symbolic and formal work, these manipulatives make algebra accessible to a broader constituency. To the stronger student, the Lab Gear offers an opportunity to gain a deeper conceptual understanding and to go beyond rote memorization of algorithms.

In grades 5-8, the Algebra Lab helps define those parts of the algebra curriculum that can be introduced as a sophisticated pre-algebra program for all students. In a first year algebra course, in 8th or 9th grade, the Algebra Lab helps introduce, illustrate, or review many concepts. It can also be used in remedial secondary classes at all levels.

The work is not easy, and you should not expect your students to pick it up spontaneously. Be sure to try the lessons on your own before trying to teach them. And if a few students are resistant to this unusual approach, be persistent—the blocks will probably grow on them.

The Algebra Lab will help you teach in three main concept areas:

- integer arithmetic
- the distributive law and factoring
- solving various types of equations

While the program does not attempt to cover the whole of algebra, these three areas cover a large part of any first-year algebra curriculum. Once they are mastered, the rest of the course is within reach.

The Lab Gear™

Because algebra is an extension of arithmetic, the Lab Gear has been designed as an extension of the most successful and effective of the manipulatives that are used to teach arithmetic—Base Ten Blocks. The Lab Gear is completely compatible with Base Ten Blocks—in fact the two can be used in conjunction with each other to teach algebra, arithmetic, or both. The use of 5-blocks and 25-blocks makes the Lab Gear more convenient to represent the generally smaller numbers that are encountered in algebra. The use of special blocks to represent two variables allows the Lab Gear to be used at a higher level of abstraction. The sizes of the x-blocks and y-blocks are carefully chosen to prevent the confusion that arises when a block (such as the Base Ten "rod") of whole number dimensions is used to represent a variable.

There are three other innovations of the Algebra Lab:

- a powerful combination of two methods to represent the minus sign (the minus area, and "upstairs")
- the workmat, for equation solving and algebraic fractions
- the corner piece, to help organize the rectangle model of multiplication and division

All of these components work together to create a unified concrete environment in which to learn the concepts of algebra.

Classroom Organization

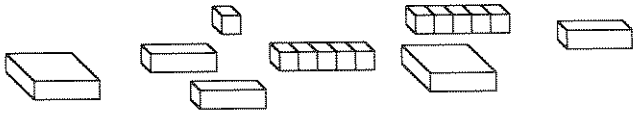
Depending on how many sets of blocks you have, organize your students in groups of three to six per tub. Within the groups, the students can work on one setup (e.g. a workmat or corner piece) individually or in pairs. In each tub, there are enough blocks for the simultaneous working out of three copies of most problems in the binder.

You probably have your own way of selecting groups. Try random groupings that change every two weeks. Random groupings are sometimes homogeneous, and sometimes heterogeneous, which allows you to take advantage of both types of arrangements. In a heterogeneous group, the stronger student can take the lead, or serve as a resource. In a homogeneous group of stronger students, there is the excitement of being able to do really fancy work fast. In a homogeneous group of weaker students, there is the comfort of not "feeling stupid" and not having to compare oneself to some star student. As long as the groupings are temporary, the students do not feel trapped, even if they do not like some of the students in the group.

Lab Etiquette

It is, of course, important that students respect the materials. A little time must be allotted at the end of each period for putting the Lab Gear back in the tub in an organized way so the next group will have no trouble finding the pieces they need. (Some group pride can be developed in relation to this.) Within their pairs, students should take turns working the problems. Beware of situations where one student does all the thinking and the other merely comes along for the ride. Pairs should periodically compare their answers and processes, just to make sure nobody is completely off-track. If differences emerge, they should try to resolve them by discussion. As a last resort, and only as a last resort, they can appeal to you for help.

Your role is to monitor the groups, acting more like a coach than a lecturer. If a group seems to lose its focus, get them back on task. If a group is too quiet, start a discussion for them. If a student is being rude to others, intervene. If an exercise is stumping an entire group, it is probably difficult for the whole class. At such moments, interrupt group work, and demand full focus on the overhead screen. Lead a class-wide discussion of the exercise. Whenever someone is speaking to the whole class, it is important for all to give their undivided attention. "No plastic in your hands" is the watchword for those times. Students who want to explain something to the group can come up to the overhead projector and demonstrate, or can talk while you demonstrate.



Using the Algebra Lab™ Lessons

The lessons in this binder are sequenced in a preview/view/review spiral. Typically, an idea is first encountered in its simplest manifestation, in the problem-solving context of an exploration. Later, a more formal lesson extends the idea into a broader range of cases. Finally, the idea gets used again as a component of more complex problems.

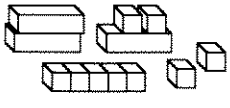
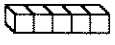
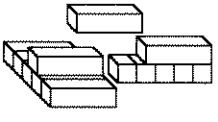
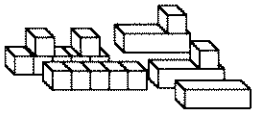
You are the best judge of which lessons are appropriate for your class. However, keep in mind that pedagogically, the most productive activities are the ones that require the students to think, experiment, and discuss, rather than merely reproduce a sequence of steps that have been demonstrated to them. A demonstration at the overhead is more effective if it follows the students' own attempts at solving a problem, rather than preceding them.

The answer is not the most important part of the problem. The process of finding it, and the concepts underlying the process are what this binder is about. Therefore, many answers are given in the body of the text so the students can check their work and verify that they are on the right track. (Or find out that they are not, and therefore need to start over or rethink their approach.) More answers are located in the back of the binder.

The classroom should not be quiet when students are using the Lab Gear. They will learn most if they are engaged in sharp discussions with each other. To keep their work organized, students should keep their own binder where pages with their work alternate with copies of the pages from this binder. Having previous pages to refer to can be invaluable to students when they are faced with a new variation on an old problem.

Students are often asked to explain their answers. This goes beyond merely "showing work," and can include writing sentences, or paragraphs, or drawing sketches. This is not easy for most students, and your coaching and coaxing is particularly important. It is worth the effort, because what they can explain they surely understand. To make sure students take this seriously, you should be sure to read what they write as often as possible. Note that explanations are not generally given in the solutions since answers will vary widely. Finally, make sure your homework assignments, quizzes, and tests include references to the work with the Lab Gear, so that the students see it as an integral part of the course.

The number of exercises in this binder should be adequate for your needs. If, however, you find that the number of exercises offered in one or another of the sections is insufficient, you can always create more yourself, or just use any algebra text as a source. But remember, it is usually preferable for the students to have a good in-depth discussion rather than to rush to do more exercises. Note that not all algebra exercises can be done with the blocks. That is also true of the exercises in this binder—not all of them can be done with the blocks. It is important for the students to remember that the Lab Gear provides a model of algebraic manipulation, but cannot encompass the flexibility of the symbolic notation that has evolved over centuries precisely because physical models proved insufficient. You will note that in many cases, the students will have to abandon the blocks in the middle of an algorithm, because, for example, there is no block to represent x/2. Conversely, you may be in a situation where the students, while working in a textbook, request access to the Lab Gear. If at all possible, you should allow them to use the blocks until they no longer need them.

Using the Explorations

In a traditional algebra class, the teacher often demonstrates how to carry out a specific algorithm, then the students are given a number of examples to practice the new skill. The problem with that approach is that often the students learn the procedures mechanically, the only justification for them being the teacher's authority, then quickly forget them.

It is in the Explorations that the Algebra Lab departs the most from this traditional approach. The Explorations include a minimum amount of instruction, relying instead on students' solving problems and constructing their own understandings of the logic that underlies the rules of algebra. The Explorations are usually quite challenging. They are designed to make such student-centered discovery possible, and are strategically placed in the binder to serve as a preview to the more formal approach used in the Lessons. If you are using the Algebra Lab in conjunction with a traditional algebra textbook, and don't have time to do everything in this binder, it is probably best to use mostly the Explorations.

The following are specific notes on the various types of Explorations:

- **Positive or Negative?** (pages 9, 23, 56, 75)
  The activities on these pages provide a way to practice simplifying expressions, and to reinforce some critical ideas about the sign of squares, cubes, and other expressions involving variables. The problems are chosen to help students think about (and hopefully avoid) some common mistakes, and clarify what happens with the sign of squares and cubes.

- **Minus Puzzles** (page 14)
  This exploration should force among the students a discussion of the correct use of the minus sign, and its representation with the Lab Gear. A good follow-up would be to ask the students to make up additional problems of this type.

- **Always, Sometimes, or Never True?** (pages 43, 53, 70, 75)
  True or False? (page 98)
  These are problems that get to the essence of the concept of variable. They are created to review or preview basic but tricky algebraic concepts and techniques, such as how one removes parentheses in different situations, or the Zero Product Principle.

  Some of the "Always, Sometimes, or Never?" problems are likely to be too difficult for your students to resolve when they are first encountered. The best approach with problems which cannot be classified on the spot is to save them until the time where the students have learned enough algebra to answer the questions themselves. It is healthy to have to deal with difficult problems, and it is important to develop a long term approach to solving them. Resist the temptation to give away the answer, though you can help the students think about it by suggesting various values for the variable.

  Student-created "Always, Sometimes, or Never?" problems (page 75) are likely to be even more difficult than the ones presented in chapters 3-5. Again, save the ones that cannot be categorized by the class until the time where they are equipped to deal with them.

- **Make a Rectangle** (pages 33, 55, 72, 95)
  Make a Square (pages 55, 95)
  These are factoring problems. They are more interesting than routine multiplications in the corner piece, because they involve more thinking, trial and error, and problem solving. In addition to factoring, they help introduce and develop the distributive rule, the main identities, and completing the square.

  Most of these activities avoid minus signs, which are reserved for the Lessons in Chapters 5 and beyond. One exception is the "Make a Rectangle" exploration on page 72, which is based on the difference of squares.

  The last activities in this series, on page 95, do not present a complete list of the blocks that are to be used. This deliberate ambiguity offers the students a final chance to pull together what they have learned in these explorations, right before the formal presentation of factoring.

- **Which is Greater?**
(pages 39, 52, 69, 81, 84)
These problems present a way to prepare students for the solving of equations and inequalities. When solving them, you may find that students want to delve deeper into the problems for which "it is impossible to tell." They may make statements like: "If x is greater than 2, then the left side is greater." Encourage them to discuss and write down these insights. They are essentially trying to solve an inequality. Questions like: "For what values of x would the two sides be equal?" lead into thinking about solving equations.

  A formal introduction to equation-solving is left until late in the book (Chapter 7). This is deliberate. Introducing this topic before students have a good feel for the meaning of algebraic symbols can lead to rote memorization of techniques with no sense of what they mean.

  When confronting the problems on page 52, your students will be tempted to do them without the blocks. If so, they will need to be extremely careful.

  On pages 69 and 81, the problems involve the use of the $\geq$ and $\leq$ symbols, which your students may not be familiar with. The first problems in these explorations are a check on the students' understanding of the effect of squaring on different numbers.

- **Area and Perimeter**
(pages 27, 38, 48, 69, 82, 96, 109, 122)
Volume and Surface Area
(pages 28, 53, 70, 82, 96, 111, 122)
The activities on these pages provide a connection with geometry, and a chance to practice combining like terms.

  To clarify the concept of surface area, the students should make paper jackets for the Lab Gear blocks, as in the example on page 28.

  If your students are getting frustrated when trying to find the perimeters of complicated Lab Gear figures, such as those on page 69, a method you could hint at is to find the perimeter of each component block. Then discuss whether the perimeter of the whole figure can be obtained by just adding the perimeters of the pieces. Of course, it cannot. Students can be guided to discover what needs to be subtracted from the sum to get the actual perimeter. The same method can be adapted to surface area calculations.

  Finding a way to continue the surface area sequences on page 70 will cause debate among your students. Accept any answer that acknowledges the repeated doubling of the volume, and the fact that the buildings are rectangular prisms.

  The perimeter and surface area sequences on pages 96, 109, and 111 turn out to be linear functions. You could have your students graph the resulting number pairs, with the position on the list as the x coordinate, and the perimeter (or surface area) as the y coordinate. The resulting graph will be a line, and can be used to predict perimeter (or surface area) for larger numbers of blocks. Another possible extension of this lesson is the introduction of subscript notation for sequences and discussion of the nth element of the sequence. Yet another extension is to try to analyze the relationship between the nature of the visual pattern and the parameters in the general formula that gives the value of the nth element. I owe the idea for this rich activity to Linda Dritsas of the Fresno, California, Education Center.

- **How Many Solutions?** (page 88)
For students to keep a sense of perspective, linear equations are introduced in the context of the questions "Always, Sometimes, or Never True?" and "How Many Solutions?" Too many students live under the erroneous belief that solving an equation always leads to the single inevitable dénouement: "x = some number."

- **Inequalities** (page 103)
This exploration should cause some vigorous discussion. Exercises 1-8 can be solved the same way that the corresponding equations would be solved. However, problems 9 and 11 should alert students to the fact that this technique does not always work. Students will probably need help answering question 12.

- **Solving for y, Solving for x and y**
(page 113)
More Solving With Two Variables
(page 114)
These explorations offer an informal introduction to solving simultaneous equations, as well as a review of basic linear techniques. They are likely to provoke lively discussions among the students.

How The Algebra Lab™ Fits into the Curriculum

The National Council of Teachers of Mathematics, and other leaders in math education are recommending:

- that algebra be introduced in elementary school
- that visual and physical models be a central part of the program
- that a problem-solving approach be used throughout math instruction
- that students work cooperatively and learn to speak and write mathematics
- that more emphasis be put on understanding, and less on complicated manipulations

The Algebra Lab incorporates all these ideas.

The Algebra Lab™: Exploring Algebra Concepts with Manipulatives can be your algebra program in grades 5-7. It should be used as part of a broader program that includes geometry, probability and statistics, and so on. In 9th grade, *The Algebra Lab: A Comprehensive Manipulative Program for Algebra I* can serve as an introduction to a full algebra course or it can be used in conjunction with any algebra textbook. It can also be a component in an integrated college preparatory curriculum. If your 8th grade students are closer to a standard 9th grade algebra class, use the high school binder. If they are not yet ready for a full-fledged algebra course, use the middle school binder. Noncollege-preparatory high school classes, can use either binder, with the goal of helping to prepare students to join the college-preparatory sequence.

If you're using this binder as part of a full Algebra I course, you should realize that work with the Lab Gear will take time. However, the extra depth of understanding gained means that you will need less time to cover the material. In general, it is best to do as much work as possible with the blocks early on in the school year, in order to establish a firm intuitive foundation to build on. However, if you will be alternating work in the Algebra Lab with work in your textbook, you need not try to match the topic sequence in the two programs. This binder has a logic of its own, and following it will mean that you will encounter some subjects earlier, and other subjects later than in your text. That is not a problem. In fact it is an opportunity for extra preview or review.

By its very nature, the Algebra Lab program uses a geometric model. Some geometric concepts are embedded in the very fabric of the program, particularly the concept of area, which is probably the most fundamental. Other geometric questions that are addressed are perimeter, volume, and surface area. This integration of another part of mathematics into the algebra program is not only a welcome breath of fresh air, but it also makes for a better understanding of the algebra concepts themselves.

A full algebra course should include more than the work with the Lab Gear. In fact, there are several important topics that cannot be addressed well with the Lab Gear. Some examples are graphing, work with exponents, work with polynomials of degree greater than two, deeply nested parentheses, "ugly" numbers, applications of algebra (except to problems of perimeter, area, and volume), and so on. At my school, we have the students work with polyominoes and pentominoes, the computer language Logo, computer graphing software, calculators, and with good problems from many sources, in addition to the Algebra Lab and a standard algebra textbook.

Please direct comments or questions about The Algebra Lab to the author, in care of Creative Publications, 788 Palomar Ave., Sunnyvale, California, 94086.

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Notes To the Student

In arithmetic, you have learned to work with numbers. Algebra is the logical extension of arithmetic. In algebra, you learn to work with symbols. Algebra is the language of all of mathematics and science. It is difficult to learn, but it is the key to so many possibilities in your life that it is worth the effort. In fact, you will probably enjoy it.

Mathematicians often use visual models to understand difficult ideas. In recent years, math teachers have been catching on to this idea, and as a result, students are learning a lot more. The Algebra Lab will help you get started in algebra. By the time you are finished with the material in this binder, you will no longer need the blocks. You will be able to do algebra the old-fashioned way, with pencil and paper.

In the future, more and more algebra will be done by computer. But what good would it do you to have a computer ready to do the algebra for you if you don't understand what algebra is? It would be as useful as a calculator to someone who didn't know the meaning of numbers.

When working with this material, I would encourage you to always have the Lab Gear blocks within reach, and to follow all the examples given with your own blocks. It is very difficult to really learn math in silence. Be prepared to discuss with your classmates and teachers any questions or ideas you may have. Finally, have a pencil and paper available to record your calculations and the answers to the exercises. The better you organize your work, the better you will understand it.

Henri Picciotto, Author