Geometry of Function Diagrams

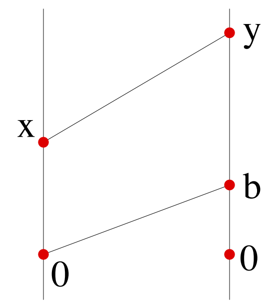
Geometry Reminders

Fact P: In a quadrilateral, if a pair of opposite sides is both equal and parallel, then the quadrilateral is a ___________. (You can prove this with the help of congruent triangles.) Opposite sides of a parallelogram are equal.

Fact T: If the corresponding angles determined by a transversal are equal, then the two lines are ___________. If two lines are parallel, then corresponding angles are __________.

Fact AA: If two pairs of corresponding angles in two triangles are equal, then the triangles are ____________.

Theorem: In the function diagram for y = mx + b, either all in-out lines are parallel, or they meet in one point



1. On the diagram of y = mx + b shown on the right, how long are the vertical sides of the quadrilateral?
2. If m=1, show that the quadrilateral is a parallelogram (and therefore that the in-out lines are parallel.)

Since x is a generic input, you have proved that in the case where m = 1, *all* in-out lines are parallel to the one through 0, and therefore to each other.

If m ≠ 1, the quadrilateral is not a parallelogram, since opposite sides are unequal, and therefore the two in-out lines meet at a point we will call F.

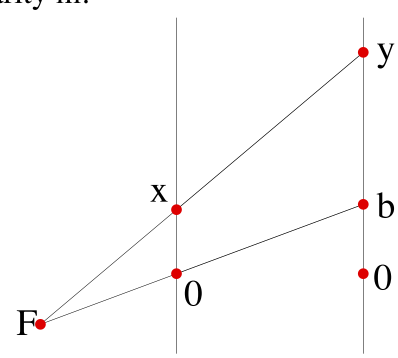
Strategy for proof: We would like to prove that *all* the in-out lines go through that same point F, the focus. To do that, we will show that the position of F on the in-out line (0, b) does not depend on the choice of x.

Consider the diagram below. Let us say that the length 0b = z. This number does not depend on x. (It depends on b, and on how wide apart the axes are.)

4. Show that the triangles Fyb and Fx0 are similar, with ratio of similarity m.

It follows that $\frac{\text{Fb}}{\text{F0}} = \text{m}$, and therefore $\frac{\text{F0} + \text{0b}}{\text{F0}} = \text{m}$

5. Use algebra to show that $\text{F0} = \frac{\text{z}}{\text{m} - 1}$ if $\text{m} \neq 1$



So F is in the same position for any input x. In other words, all the in-out lines go through the focus, which is what we wanted to prove.

Theorem: If a function diagram has a focus, then the equation is of the form y = mx + b

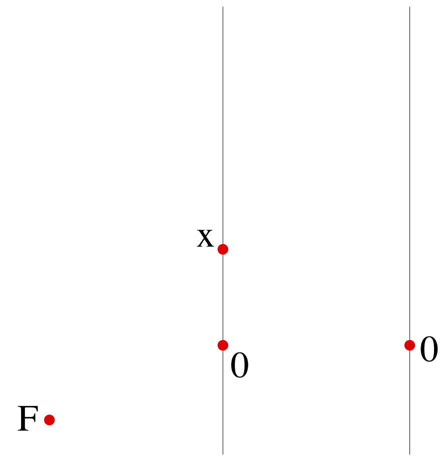
In other words, if all the in-out lines of a function diagram meet in one point F, then the image of all points x is given by the same formula, in the form y=mx+b.

Strategy for proof: we will use similar triangles to help us find a formula for the output corresponding to the input x.

1. Using a ruler, draw in-out lines for 0 and x. Label the output for 0 as b, and the output for x as y.

Call the ratio Fb / F0 = m.

2. Show that the two triangles are similar, with ratio m.
3. Find the ratio of the vertical sides, and solve for y.



Since m and b are constants that depend only on F's position, and since x was a completely generic point, we have proved the theorem.

Theorem: If all the in-out lines of a function diagram are parallel, then the equation is of the form y = x + b

In other words, if all the in-out lines are parallel, then the image of all points x is given by the same formula, in the form y=x+b.

Strategy for proof: we will use a property of parallelograms to help us find a formula for the output corresponding to the input x.

4. Why is the quadrilateral a parallelogram?
5. Use a property of parallelograms to show that y = x + b

Since b does not depend on x, and since x is a completely generic point, we have proved the theorem.