The McNuggets Problem

One day you go to McDonalds and try to order 13 McNuggets. You are told that is impossible. One can only order 6, 9, or 20 nuggets. You could get 12 (because that's 6+6) or 15 (because that's 6+9), but not 13.

What numbers of McNuggets is it possible to order? What numbers of McNuggets are impossible to order? (Assume you must include all the McNuggets you order in the count)

Lesson Plan

Central Question

What is the largest number of McNuggets that you *cannot* buy?

Prerequisites
There are few prerequisites for this lesson, as it is accessible to any student who can add.

Lesson Overview
McDonald's sells chicken McNuggets in boxes of 6, 9, and 20. As a result, it is possible to purchase certain numbers of nuggets, and impossible to purchase other numbers. Investigating these numbers becomes an interesting mathematical puzzle when one realizes that--after a certain number--one can purchase any number of nuggets. How can you find that number? An extension to this problem is included, which encourages the student to generalize this question, focusing on just two starting numbers.

1. Question
Begin by telling a story about visiting McDonalds and wanting to order exactly 13 McNuggets, initiating the problem orally, then passing out the *McNuggets Problem* handout later. Or, give the handout immediately. Regardless, the following problem is established: If McNuggets are sold in boxes of 6, 9, and 20, then can I order exactly 13 nuggets? Why not? What numbers of nuggets can I order? What number of nuggets cannot be ordered?

2. Investigation
Students should work in small groups (four tends to be optimal) and encouraged to attack the above questions with little initial guidance, so as to promote student creativity. Say something like, ``It looks like we may have a lot of numbers to keep track of. You should decide on a system to organize your investigation."

At this point, there is ample opportunity for discussion. Things that come up may include:

- It's obvious that you cannot buy some certain small numbers (e.g., 4), but after some point, can you actually buy any number of nuggets?
- What is the largest number of McNuggets that you *cannot* buy?

Hint 1: You may decide to directly state that indeed such a number exists.

- Is there a systematic way to find this number?

You will need to decide how much time to give students to work on this.

Hint 2: Pass out the *McNuggets Chart*. Pass this out without saying anything more than, ``This chart may help you organize your thinking." Note that this chart is organized so that any number directly below another is obtained by adding six.

3. Summary, Generalization, and Extension

The chart helps to show that there is indeed a largest number that cannot be purchased, because once a student circles a number, all numbers (forever!) below the circled solution can also be purchased. (Because one can always add six more nuggets to any given number of purchased nuggets...by buying one more box of six.) See the *Wrap-Up* for more details.

Through small group and whole class discussion, the teacher should guide the class to understand that 43 is the largest number of McNuggets that cannot be purchased. Ask students to write down and share their reasoning, explaining why the chart helped.

Suggested Extension

You may want to extend and generalize the McNuggets Problem, most likely on another day. If so, use the *McNuggets Extension*.

McNuggets Chart

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78

McNuggets Extension

Recall *The McNuggets Problem.* As a follow-up, let's generalize it. To make this manageable, we will start with two numbers, not three.

1. You have an unlimited supply of dimes (10 cents) and quarters (25 cents). What amounts can be obtained, and what amounts cannot be obtained by combining them?

2. At Fred's Kitchen Supply, cabinets are available in two lengths: 3 feet and 5 feet. By putting cabinets end to end, Fred can accommodate walls of different lengths. Imagining that kitchens can be arbitrarily large, what length walls are possible to line exactly with cabinets? What lengths are impossible?

3. What numbers can be obtained by adding the numbers 6 and 9 as many times as you want? What numbers cannot be obtained?

4. In 1958, it cost 4 cents to mail a letter in the United States. In 1963, it cost 5 cents. Imagine you have an unlimited supply of 4 and 5-cent stamps. What amounts can you make? What is the largest amount you cannot make?

Generalizing

5. By now, you are probably aware that some pairs of building block numbers work better than others.
   a. Which types of pairs allow us to build every number beyond a certain point?
   b. For the other pairs, what do they allow us to build?

6. Let us use p and q as our numbers, with $p < q$. To find the largest impossible number, use the strategy we used for the McNuggets.
   Organize numbers in p columns.
   Circle p and all the numbers below it.
   Circle q and all the numbers below it.
   Circle 2q and all the numbers below it.
   Circle 3q and all the numbers below it.
   …
   Continue until you have reached all the columns.

7. Explain why this is true:
   a. The last multiple of q you circled is $(p - 1)q$.
   b. The last uncircled number is $(p - 1)q - p = pq - (p+q)$.

Reference

Solution to *The McNuggets Problem*

By adding 6 repeatedly to numbers that are possible, one gets more numbers that are possible. This shows up in the chart as a vertical column of possible numbers

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72

But we can also add 9 to numbers that are possible. So we can circle 20+9 and all the numbers below it.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72

And finally, we can circle 20+20, and 20+20+9, and all the numbers below them:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72

That does it. Conclusion: all numbers beyond 43 are possible!

Solution to *McNuggets Extension*

To generalize the McNuggets Problem, it is best to start with just two numbers. That is, what combination of two start-up numbers allow us to reach every number beyond a certain one? It turns out that the two starting numbers must be relatively prime (i.e., they must have no common divisors other than 1). This should be clear after working on #1-5. Notice that the three numbers (taken together, not pair-wise) in the McNuggets problem are indeed relatively prime.

When starting from a combination of two relatively prime numbers, the key is to organize numbers in columns, as we did for the original problem. Use the smaller of the two numbers as the number of columns. Once numbers are organized this way, one can show that the largest integer that is not the sum of multiples of p and q turns out to be pq-p-q*, or the product minus the sum*. This is easiest to see by looking at a specific example, such as a=5 and b=7, following the strategy in #6-7, and discussing how the strategy plays out in the example.

Real World Connection

This problem is a modern version of a classical number theory problem from the mathematical folklore. Older versions are known as ``The Postage Stamp Problem" and ``The Coin Problem". See http://en.wikipedia.org/wiki/Coin_problem. These problems are known as Frobenius problems, named after the number theorist Ferdinard Frobenius. The solutions create what is called a numerical semigroup, which also have applications in a field of mathematics known as algebraic geometry. Students who find the McNuggets Problem to be worthwhile may be interested in studying number theory in the future. An excellent resource that exposes students to the life of a pure mathematician--and specifically number theorists--is the Nova special *The Proof* (http://www.pbs.org/wgbh/nova/proof/.

Source

Algebra: Themes, Tools, Concepts by Anita Wah and Henri Picciotto
(available at www.MathEducationPage.org/attc)