LESSON 8.11

Negative Bases, Negative Exponents

RECIPROCALS

In previous lessons, we have considered only whole number exponents. Does a negative exponent have any meaning? To answer this, consider these patterns.

$3^4 = 81$	$(1/3)^4 = 1/81$
$3^3 = 27$	$(1/3)^3 = 1/27$
$3^2 = 9$	$(1/3)^2 = 1/9$
$3^1 = 3$	$(1/3)^1 = 1/3$
$3^0 = 1$	$(1/3)^0 = 1$
$3^{-1} = ?$	$(1/3)^{-1} = ?$

1. a. Look at the powers of 3. How is each number related to the number above it? Following this pattern, what should the value of 3^{-1} be?

b. Now look for a pattern in the powers of 1/3. As the exponent increases, does the value of the power increase or decrease? Following this pattern, what should the value of $(1/3)^{-1}$ be?

c. Compare the values of 3^{-1}, 3^1, $(1/3)^1$ and $(1/3)^{-1}$. How are they related?

d. Use the pattern you found to extend the table down to 3^{-4} and $(1/3)^{-4}$.

Another way to figure out the meaning of negative exponents is to use the product of powers law. For example, to figure out the meaning of 3^{-1}, note that:

$$3^{-1} \cdot 3^2 = 3^1$$
$$3^{-1} \cdot 9 = 3$$

But the only number that can be multiplied by 9 to get 3 is 1/3, so 3^{-1} must equal 1/3.

2. Find the value of 3^{-1} by applying the product of powers law to $3^1 \cdot 3^{-1}$.

3. Use the same logic to find the value of:
a. 3^{-2}; b. 3^{-x}.

4. Are the answers you found in problem 3 consistent with the pattern you found in problem 1? Explain.

5. **Summary** People who have not studied algebra (and, unfortunately, many who have) think that 5^{-2} equals a negative number, such as -25.

a. Write a convincing argument using the product of powers law to explain why this is not true.

b. Show how to find the value of 5^{-2} using a pattern like the one in problem 1.

6. a. Show that $5x^2$ and $5x^{-2}$ are not reciprocals, by showing that their product is not 1.

b. Find the reciprocal of $5x^2$.

MORE ON EXPONENTIAL GROWTH

A bacterial culture doubles every hour. At this moment it weighs 10 grams.

7. What did it weigh
a. 1 hour ago?
b. 2 hours ago?
c. x hours ago?

8.

a. Explain why the weight of the bacteria culture x hours from now is given by
$$W = 10 \cdot 2^x.$$

b. Explain the meaning of substituting a negative value for x.

9. Show your calculations, using the equation in problem 8, to find out:
   a. how much it will weigh in three hours;
   b. how much it weighed three hours ago.

In 1975 the world population was about 4.01 billion and growing at the rate of 2% per year.

10. If it continued to grow at that rate, write a formula for the world population after x years.

If it had been growing at the same rate before 1975, we could estimate the population in previous years by using negative values of x in the formula.

11. Use your calculator to find the value of $(1.02)^4$ and its reciprocal, $(1.02)^{-4}$.

12. Show your calculations using the equation in problem 10 to estimate the population in:
    a. 1971; b. 1979.

13. Assume the world population had been growing at this rate since 1925.
    a. Estimate the world population in 1925.
    b. Compare this number with the actual world population in 1925, which was about 2 billion. Was the population growth rate between 1925 and 1975 more or less than 2%? Explain.

RATIO OF POWERS

Negative exponents often arise when simplifying ratios of monomials.

This law of exponents is sometimes called the *ratio of powers* law:

$$\frac{x^a}{x^b} = x^{a-b}\text{, as long as } x \text{ is not } 0.$$

However, notice that it works only when the bases are the same.

Examples:

$$\frac{x^6}{x^7} = x^{6-7} = x^{-1} \text{ or } \frac{1}{x^1}$$

$$\frac{x^{3a}}{x^{5a}} = x^{3a-5a} = x^{-2a} \text{ or } \frac{1}{x^{2a}}$$

14. Simplify.
    a. $4x^6/5x^7$ b. $2x^8y^3/2xy$
    c. y^3/y^7 d. $45a/9a^5$

15. Simplify these ratios.
    a. $\frac{400a^5}{25a^2}$ b. $\frac{400x^3}{200x^8}$
    c. $\frac{3m^6}{9m^3}$ d. $\frac{9R^a}{3R^a}$

16.
    a. Write as a power of 4, $4^{3+x}/4^{3-x}$.
    b. Write as a power of 7, $7^{5x-5}/7^{5x-6}$.

17. Solve for x.
    a. $\frac{7^4}{7^{x+2}} = 7^3$
    b. $\frac{3 \cdot 5^{x+2}}{12 \cdot 5^2} = \frac{1}{20}$

18. Divide without using your calculator. Then, if your answer is not already in scientific notation, convert it to scientific notation.
    a. $\frac{4.2 \cdot 10^5}{3.0 \cdot 10^2}$ b. $\frac{3.0 \cdot 10^4}{1.5 \cdot 10^6}$
    c. $\frac{1.5 \cdot 10^3}{3.0 \cdot 10^6}$ d. $\frac{9 \cdot 10^a}{3 \cdot 10^b}$

OPPOSITES

The expression $(-5)^3$ has a negative base. This expression means *raise -5 to the third power.* The expression -5^3 has a positive base. This expression means *raise 5 to the third power and take the opposite of the result.*

19. Which of these expressions represent negative numbers? Show the calculations or explain the reasoning leading to your conclusions.

-5^3 $(-5)^3$ -5^2 $(-7)^{15}$ $(-7)^{14}$

-5^{-3} $(-5)^{-3}$ -5^{-2} $(-7)^{-15}$ $(-7)^{-14}$

20.

a. Is $(-5)^n$ always, sometimes, or never the opposite of 5^n? Explain, using examples.

b. Is -5^n always, sometimes, or never the opposite of 5^n? Explain, using examples.

EARLY PAPERS

Ms. Kem has a policy that penalizes students for turning in papers late. Her students are trying to convince her to give them extra points for turning in their papers early. Some students propose a policy based on adding points. Others propose one based on increasing by a percentage.

21. If you were her student, what kind of early paper policy would you propose?

22. Using your policy, what would your score be, if your paper were x days early?

REVIEW *WHICH IS GREATER?*

Or are they equal?

23. a. $x - 0.30x$ b. $0.70x$

24. a. $(0.70)(0.70)x$ b. $x - 0.50x$

25. a. $(0.90)(0.90)(0.90)x$

b. $x - 0.10x - 0.10x - 0.10x$

REVIEW *EQUATION SOLVING*

Solve for x.

26. a. $(0.85)(0.85)(0.85)(0.85)x = 18.79$

b. $x - 0.2x = 160$

c. $0.80x = 500$

27. $\frac{50b^3}{xb} = 2b^2$

28. $\frac{20a^{m+1}}{10a^m} = 2a^x$

REVIEW *WHAT'S THE FUNCTION?*

29. Find the slope of the line that goes through each pair of points. Then find the equation for the line. (Hint: A sketch may help.)

a. (0, 1) and (2, 3)

b. (0, 4) and (0.5, -6)

c. (0, 7) and (-0.8, 0.9)

30. In problem 29

a. how did you find the y-intercept?

b. how did you find the slope?

31. Find the equation for the line

a. having slope 0.9, passing through (2, -1);

b. having slope 3.4, passing through (6.7, 9);

c. passing through (8, 2) and (1.3, -5.4).