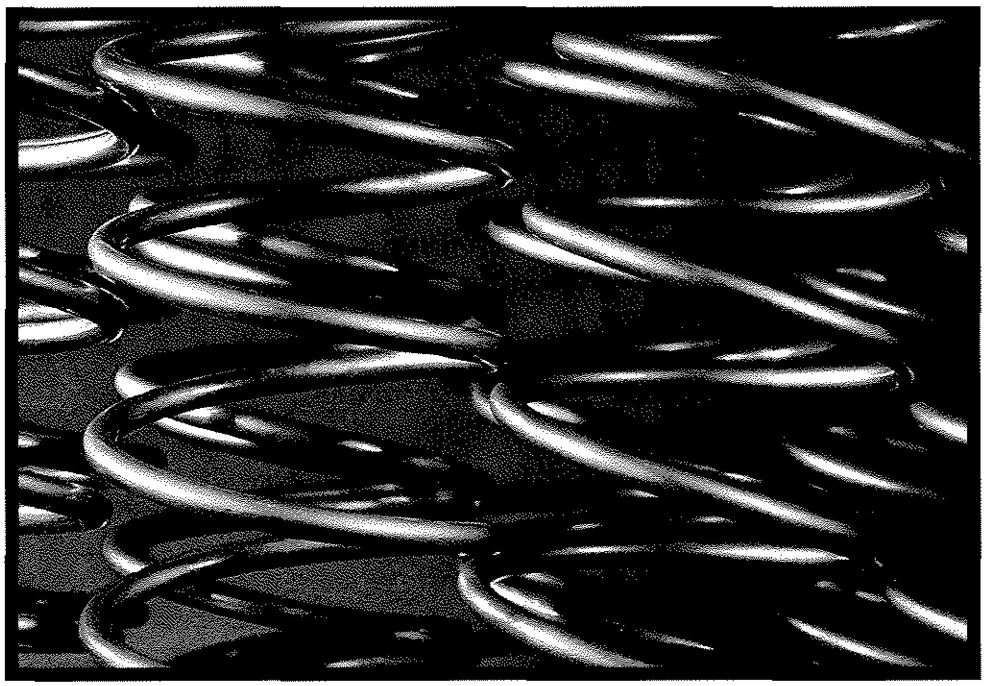
CHAPTER 12



The spiral coils of car springs

Coming in this chapter:

Exploration By measuring people's feet and asking for their shoe size, find a formula relating foot length, in inches, to shoe size,

a. for men;
b. for women.

MATHEMATICAL MODELING

LESSON 12.1

The U.S. Population, 1890–1990

You will need:

graph paper

colored pens

The Bureau of the Census conducts a census every ten years, as required by the U.S. Constitution. Census results are now used for many purposes, but their original purpose was primarily to determine how many seats each state would be allocated in the House of Representatives. As population patterns change, these seats are divided up differently among the states. Here are some census results from 1890 through 1990.

Census Table

Year	Population	# increase	% increase
1890	62,979,766	12,790,557	25.5
1900	76,212,168	13,232,402	21.0
1910	92,228,496	16,016,328	21.0
1920	106,021,537	13,793,041	15.0
1930	123,202,624	17,181,087	16.2
1940	132,164,569	8,961,945	7.3
1950	151,325,798	19,161,229	14.5
1960	179,323,175	27,997,377	18.5
1970	203,302,031	23,978,856	13.4
1980	226,545,805	23,243,774	11.4
1990	248,709,873	22,164,068	9.8

EXAMINING DATA

1. Use the information given to estimate the 1880 population.
2. Write a paragraph describing anything interesting you see in the data. What factors affect population growth? Can you think of historical events that might be associated with periods of low or high growth rates?
3. Over what ten-year period was
   a. the number increase the smallest? The largest?
   b. the percent increase the smallest? The largest?
4. From 1890 to 1990, what was the overall
   a. number increase?
   b. percent increase?
5. Using a large piece of graph paper, make a graph of the population as a function of time. You will add to this graph when you do other problems in this lesson. Choose the scale carefully.
6. Approximately when did the population reach its halfway point between 1890 and 1990? Explain how you calculated this, and show the point on your graph. Is the halfway point in population before or after the halfway point in years?

MAKING PREDICTIONS

If you had lived in 1890 and wanted to predict the population of the United States in 1900 and 1990, you might have assumed that the increase for every future ten-year period would be the same as it had been in the ten-year period from 1880 to 1890. (We will call 1880-1890 the *base period.*)

7. For each assumption below, make a table showing what the population would have been in each decade.
   a. The number increase in each decade was the same as in the base period.
   b. The percent increase in each decade was the same as in the base period.

8. In this problem, add to the graph you made in problem 5. Use a different color for each set of data.
   a. Graph the data in your table from problem 7a. Write a description on your graph telling what assumption was used to obtain the numbers.
   b. Repeat part (a) for the data in your table in problem 7b.

9. Compare the three graphs.
   a. Which of the two assumptions in problem 7 gave a closer prediction of the population in 1900? How close was each estimate?
   b. Which predicted the population in 1990 more accurately? How close was each estimate?

Say you were living in 1940, had access to the data for the period 1890-1940, and wanted to predict the population for 1950 and 1990.

10. a. Why might you *not* want to use the growth from 1930 to 1940 to help you make the predictions?
    b. What numbers might you choose instead to model a constant number increase? What about a constant percent increase?

11. Repeat problem 7, starting with the 1940 population and using the numbers you chose in problem 10. Do you get better predictions?

12. Predict the population of the U.S. in the years 2000 and 2040. Explain how you arrive at your numbers.

ESTIMATING MISSING DATA

It is very expensive to conduct a census of the entire population, and it cannot be done every year. However, census data can be used to estimate the population in other years.

13. Use the 1940 and 1950 data to estimate the population in 1945 assuming
    a. linear growth;
    b. exponential growth.

14. Use the 1930 and 1950 data to estimate the population in 1940 assuming
    a. linear growth;
    b. exponential growth.

15. Use the 1890 and 1990 data to estimate the population in 1940 assuming
    a. linear growth;
    b. exponential growth.

16. Compare your answers to problems 14 and 15. Did you get closer to the actual 1940 population using
    a. the 1930 and 1950 data or the 1890 and 1990 data?
    b. the linear model or the exponential model?

EXAMINING ASSUMPTIONS

Definitions: When we know data points and use them to determine data values between those points, the process is called *interpolation*. When we know data points and try to use them to predict data values at a later or earlier time, the process is called *extrapolation*.

17. Which of the problems in this lesson involved extrapolation? Which ones involved interpolation?

It is important to examine assumptions when analyzing from data. People who analyze data often make incorrect projections and draw wrong conclusions because of making inappropriate assumptions.

18. **Report** Write a report summarizing what you learned in this lesson. Your report should include but not be limited to comments on:

- the suitability of the linear and exponential models as applied to the growth of the U.S. population during this century;
- the validity of results from extrapolating and interpolating using these models;
- a comparison of the accuracy of short-term and long-term predictions;
- how ideas outside of mathematics can help improve the quality of a mathematical model.

REVIEW LINE THROUGH TWO POINTS

19. Find the equation of the line through:

a. (0, 0) and (12, 34);

b. (5, 6) and (7, 11);

c. (8.9, -10) and (12.3, -4.3).

LESSON 12.2

The Median-Median Line

You will need:

graph paper

FITTING A LINE

The table shows fuel efficiency data for 28 automobiles equipped with manual transmission.

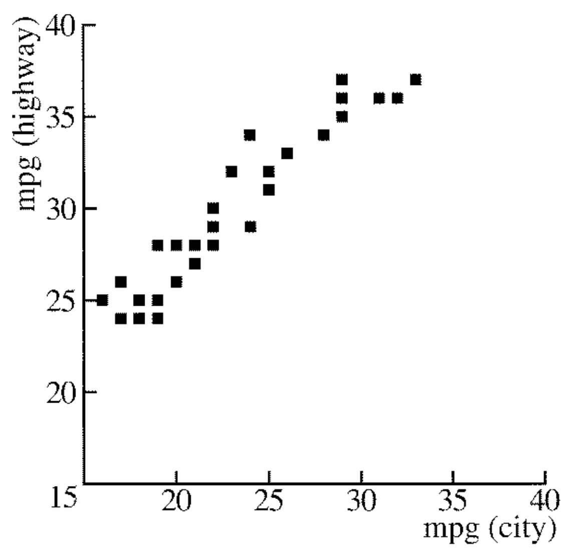
Highway vs. City Mileage

EPA Fuel Efficiency Data for 28 Cars with Manual Transmission

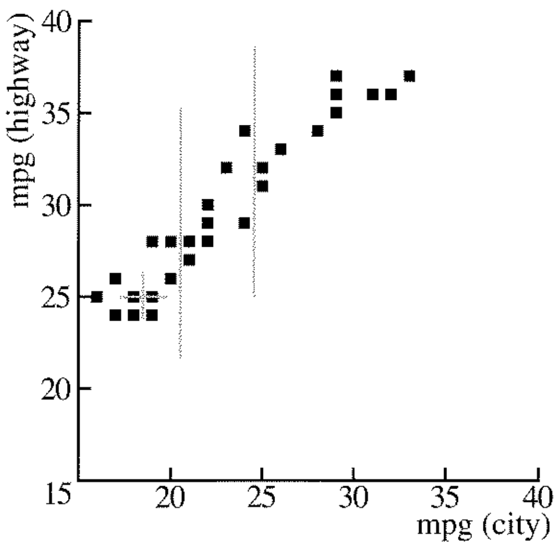
Car	miles per gallon	
	City	**Highway**
Corvette	16	25
Firebird	17	26
Thunderbird	17	24
Nissan 300ZX	18	24
Subaru XT	18	25
Stealth	19	24
Saab 9000	19	25
Sunbird	19	28
Volvo 740	20	26
Shadow	20	28
Probe	21	27
Sonata	21	28
Nissan NX	22	28
Colt Vista	22	29
Celica	22	30
Eclipse	23	32
Accord	24	29
Acclaim	24	34
Capri	25	31
Cabriolet	25	32
Impulse	26	33
Geo Prizm	28	34
Colt 5-speed	29	35
Escort	29	36
Sentra	29	37
Colt 4-speed	31	36
Civic CRX	32	36
Tercel	33	37

1. Explain the meaning of the words *city mileage* and *highway mileage*.

Can average highway mileage be predicted from average city mileage? A graph of highway mileage versus city mileage shows that the data points lie approximately in a straight line. In this lesson you will learn a formal method for fitting a line to data. You can then use this line to make predictions for other cars.



2. On your own graph paper, make a full-page graph of the data. Your graph should have both scales starting at (0, 0). Use vertical lines to divide the data points into three approximately equal sets of points, as shown in the following graph. There are ten points in the first set, eight in the middle set, and ten in the third set.



Look at the first set of data points. In your table, this is (16, 25) through (20, 28). It is easy to see on the graph that the median of the x-values in this first set of points is 18.5, and the median of the y-values is 25. The median point is marked with a +. Five points are to the left of it and five to the right. Five points are below it, (or even with it), and five are above, (or even with it).

3. Plot the point (18.5, 25) on your graph to show the medians of the x-values and y-values. Mark it with a +.

4. Find the median of the x-values and the median of the y-values for the second set of points. Mark it with a +.

5. Repeat for the third set of points.

The three +'s do not all lie exactly on the same line, but we can find a line that is close to all of them.

6. Place your ruler next to the first + and the third +, as if you were going to connect them with a line, but do not draw a line. Instead, move your ruler slightly toward the second +, about one-third of the way. Then draw the line.

7. Using two points on the line, find its equation. (Use points on the line, not actual data points — unless they happen to lie on the line.)

The line for which you found the equation is called the *median-median* line. Its equation provides an approximate relationship between city and highway mileage for a given car.

EXAMINING THE MODEL

Summary

8. What is the slope of the fitted line? What is its meaning in terms of this application?

9. What is the y-intercept of your line? What is its meaning in terms of this application?

10. Find two data points that are at least two units above the fitted line. What cars do they represent? What does it mean for points to be above the fitted line?

11. Find two data points that are at least two units below the fitted line. What cars do they represent? What does it mean for points to be below the fitted line?

12. Find two data points that are exactly on the line, or very near it. What cars do they represent? What does it mean for points to be on or near the fitted line?

USING THE MODEL

13. Using your model (the equation of your fitted line), predict the highway mileage for a car that got city mileage of:

a. 30 miles per gallon;

b. 27 miles per gallon.

14. For a city mileage of 26, what is the

a. actual highway mileage based on the data?

b. predicted highway mileage based on the fitted line?

15. For a highway mileage of 28,

a. what range of city mileages might you expect, based on the data?

b. what city mileage would you expect, based on the fitted line?

EXTENDING THE MODEL

16. Use the equation of the fitted line to predict highway mileage, if the city mileage is the following:

a. 53 b. 11

17. Based on your model, what city mileage would you expect for highway mileage of:

a. 58? b. 15?

Car	miles per gallon	
	City	**Highway**
Lamborghini	9	14
Ferrari	10	15
BMW M5	11	20
Suzuki Swift	45	50
Civic HF	49	52
Geo Metro	53	58

This table shows data for cars with very high and very low mileage.

18.

a. Does your model seem to work for very high and very low values?

b. For what range of values does your model work well? Explain.

YOUR OWN DATA

19. Project Collect your own data (at least twenty *pairs* of numbers), either from an almanac, newspaper, or magazine, or by surveying people you know. Graph the data. *If the points seem to fall more or less in a line*, find the median-median line and find an equation for it. In any case, write a paragraph about what you find out. The following are possible topics, but you may choose any two variables which are related.

a. arm span vs. height

b. weight vs. height

c. height vs. shoe size

d. points scored vs. time on the court

e. hits vs. times at bat

LESSON 12.3

Safe Driving

You will need:

graph paper

There is no safe way to drive after drinking. Alcohol reaches a person's brain very soon after it has been absorbed into the bloodstream, and it impairs vision, hearing, muscular coordination, judgment, and self-control.

A person can begin to show mild effects from drinking alcohol when the blood alcohol concentration (BAC) is as low as 0.02%. Most people do not experience impairment until the BAC is about 0.05%, but each situation is different. A person who is tired or sick, or has taken drugs or medicines, may experience impairment with a lower BAC. In any case, a BAC of 0.10% is very unsafe for driving.

A FORMULA

Blood alcohol concentration depends on many factors, but it can be estimated by using a person's weight and the amount of alcohol consumed, using this formula.

$$B = \frac{7.6 \cdot A}{W}$$

$\boldsymbol{B}$ = blood alcohol level, or BAC (in %).
$\boldsymbol{A}$ = alcohol consumed (in ounces).
$\boldsymbol{W}$ = body weight (in pounds).

The number 7.6 in the formula was derived by taking into account physiological factors (such as the percentage of alcohol that will be absorbed into the blood) and conversion of units.

Definition: We say that y is *inversely proportional* to x if the product of x and y is constant. Expressed algebraically
$xy = k$ or $y = k/x$ for some constant k.

1. In the formula is B
   a. directly or inversely proportional to W?
   b. directly or inversely proportional to A?

2. Use the formula to estimate the blood alcohol concentration of:
   a. a 152-pound person who consumed one ounce of alcohol;
   b. a 190-pound person who consumed two ounces of alcohol.

3. a. Solve the formula for W in terms of the other two variables.
   b. Use your equation to estimate the weight a person would have to be in order to have a blood alcohol concentration of 0.05 after drinking three ounces of alcohol.

4. a. Solve the formula for A in terms of the other two variables.
   b. Estimate the amount of alcohol a person probably consumed if he or she weighed 170 lbs. and had a BAC of 0.10.

GRAPHING BAC vs. ALCOHOL

The formula has three variables, so we cannot graph it on a two-dimensional Cartesian coordinate system. However, we can use two-dimensional graphs to study this problem by fixing the value of one variable and graphing the resulting function.

5. a. Substitute 152 for W in the formula to find the function that expresses how BAC depends on the amount of alcohol consumed for a 152-pound person.
   b. Make a graph of the function you wrote in part (a). Label the y-axis *BAC (%)* and the x-axis *Alcohol (oz)*.
   c. Label your graph so people can see what it refers to.

6. Repeat problem 5 for three other reasonable weights. Use the same axes for all four graphs.

7. Describe the four graphs you drew. For a given body weight, is BAC directly proportional or inversely proportional to the amount of alcohol consumed? Explain.

GRAPHING BAC vs. WEIGHT

8. a. Substitute 1 for A in the formula to find the function that expresses how BAC depends on weight for people who have consumed one ounce of alcohol.
   b. Make and label the graph of the function you wrote in part (a).

9. Repeat problem 8 for three other amounts of alcohol (between two ounces and eight ounces). Use the same axes for all four graphs.

10. Describe the four graphs you drew. For a given amount of alcohol, does the BAC vary directly or inversely as the weight of the person? Explain.

THE EFFECT OF TIME

Alcohol does not stay in a person's blood forever. The amount remaining as time passes depends on many factors. A rule of thumb is that 40 minutes after drinking, blood alcohol starts to decrease at the rate of 0.01% per 40 minutes. In using this rule, you must remember to account for the first 40 minutes.

Example: Using the formula, a woman's blood alcohol level was estimated to be about 0.06%. Eighty minutes later she might expect it to be about 0.05%.

11. A man's blood alcohol concentration was estimated to be about 0.09%. How long would he have to wait for his BAC to drop below 0.02%?

12. A woman's blood alcohol concentration was estimated to be about 0.12%. How long until her BAC was below 0.04%?

SAFE DRIVING GUIDELINES

13. A 115-pound woman had two ounces of alcohol to drink. Her 240-pound companion drank three ounces. Two hours later, do you think either person could drive safely? If so, which one? Explain your answer.

14. People know how much they have had to drink, but they do not know how much alcohol they have consumed. Calculate the amount of alcohol in each of these drinks.
    a. 12 ounces of beer that is 4% alcohol
    b. 4 ounces of wine that is 12% alcohol
    c. 6 ounces of wine that is 12% alcohol
    d. 4 ounces of a drink that is 20% alcohol

15. A woman drank two 12-ounce beers. She weighs about 120 pounds. How long should she wait before driving? Explain.

16. **Report** Write a report that will give information to people to help them use good judgment in driving if they have been drinking. Include the following components in your report:
    - Summarize what you learned about blood alcohol concentration in your investigation. You may wish to include graphs or tables.

- Make a chart or diagram that you think will help give people information about blood alcohol concentration. They should be able to look up their weight and the amount they have had to drink in your table and estimate their BAC. Include information on the amount of alcohol in some typical drinks.

17. Research Find out about the DUI (driving under the influence of alcohol or drugs) laws in your state. In some states the laws are different for people under age 18 or 21. You may want to find statistics about the relationship between BAC and the chance of being involved in an accident.

- Summarize what you find out about DUI laws.
- Give your own opinion about the DUI laws in your state.

REVIEW *COMPARING FRACTIONS*

Explorations

18. Find several values of x for which:

a. $\frac{x}{40} > \frac{40}{x}$ b. $\frac{x}{40} < \frac{40}{x}$

19. Which is greater?

a. $\frac{x}{40}$ or $\frac{x}{45}$ b. $\frac{40}{x}$ or $\frac{45}{x}$

20. Which is greater?

a. $\frac{d}{40} - \frac{d}{45}$ or $\frac{d}{40} - \frac{d}{50}$

b. $\frac{d}{40} - \frac{d}{45}$ or $\frac{d}{45} - \frac{d}{50}$

LESSON 12.4

Mathematical Models in Science

You will need:

- graph paper
- materials for experiments (See below)

HEATING AND COOLING GASES

Doing science often means finding mathematical models that fit experimental data. In 1787 the French scientist Jacques Charles discovered that when a gas is kept at constant pressure, it expands when heated and contracts when cooled. For gases under constant pressure, *volume is a linear function of temperature*.

At a certain pressure a gas has a volume of 500 cubic centimeters at 27°C. Kept at the same pressure, it expands to 605cc at 90°C.

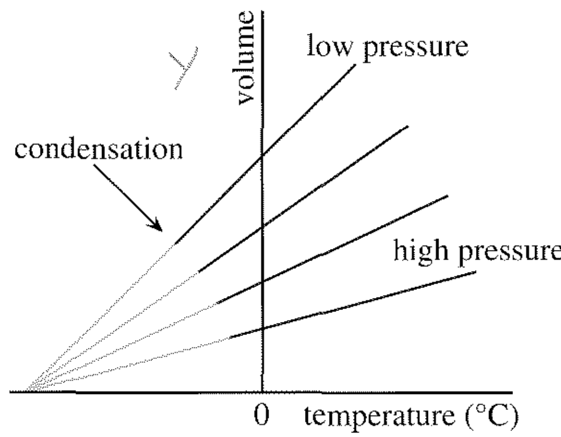
1. Find an equation that gives the volume (V) of this gas as a linear function of the temperature (T).
2. Find the volume of the gas at 0°C.
3. When kept at this pressure, how much does this gas expand for every 1°C increase in temperature?

When they get cold enough, gases condense (turn into liquids). If they did not, the temperature for which the volume would be 0 is called *absolute zero*, the lowest possible temperature.

4. Use the equation you wrote in problem 1 to figure out what temperature absolute zero must be.

This graph shows how the volume of a certain gas varies with temperature, when kept at constant pressure. Each line represents a different pressure. The point where the red line ends and the blue line starts is the condensation point. Only the red lines represent actual data.

Charles's Law

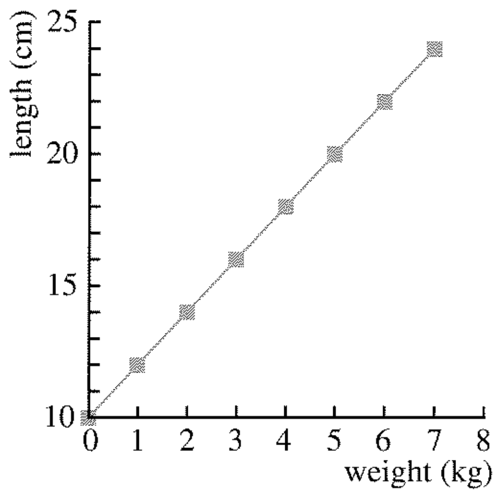


Summary

5. As the pressure increases, what happens to the slopes of the lines? What does this mean in terms of the application?
6. What is the meaning of the y-intercept of the lines? As the pressure increases, how does it change?
7. How does the condensation point vary with pressure?
8. Why do all the blue lines intersect at one point? What is the point's significance?

STRETCHING IT

The length of a spring is related to the weight that hangs from it. The following figure shows a graph from an experiment with a certain spring.



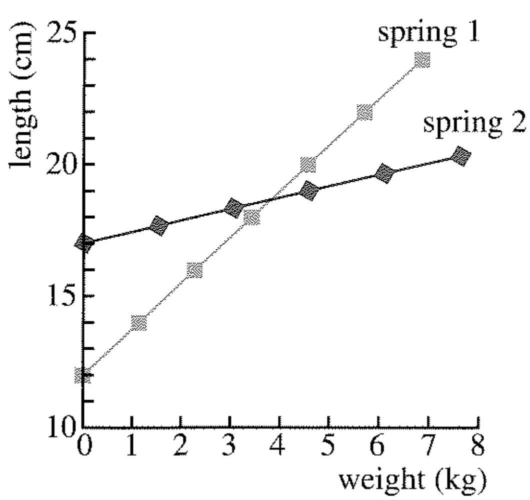
9. a. What was the length of the spring before any weight was added?

b. How many centimeters did the spring stretch for each kilogram of weight?

c. What is the equation that relates length to weight?

10. Can the graph be indefinitely extended to the right? Explain.

11. This graph shows data for two other springs. Which spring is stiffer? Which one is longer? Explain.



DEPENDENCE AND INDEPENDENCE

Paul, a forest lover, knew how to estimate the temperature from listening to cricket chirps. His grandmother had taught him to count the number of chirps per minute, divide by 4, and add 40 to get the temperature in Fahrenheit.

12. Write an equation for the temperature (T) as a function of the number of chirps (C).

13. What would Paul estimate the temperature to be if he counted 180 chirps per minute?

14. According to the model, at what temperature would the crickets cease to chirp?

Since the number of chirps depends on the temperature, and not vice-versa, we call number of chirps the *dependent* variable and temperature the *independent* variable.

In algebra we usually call the independent variable x and use the horizontal axis for it. We call the dependent variable y and use the vertical axis for it. Likewise, we often express the relationship between the two variables by writing the dependent variable as a function of the independent variable.

15. Write an equation for the dependent variable (number of chirps) as a function of the independent variable (temperature). Hint: Use your equation from problem 12 and solve for C in terms of T.

In an experiment the independent variable is the variable we change or manipulate. Then we observe and record the effect on the dependent variable.

16. Which variable is dependent and which is independent in problem 9? Explain.

YOUR OWN EXPERIMENTS

For each experiment, problems 17-20, do the following:

a. Discuss the relationship you expect between the two variables.
b. Identify the dependent and independent variables.
c. Carry out the experiment and collect the data in a table.
d. Make a graph.
e. Interpret the graph.
f. If possible, write an equation relating the variables.
g. Draw some conclusions.

17. *Spring:* The length of a spring as a function of the weight that hangs from it — **You will need** a spring and several identical weights. Start by letting the spring hang freely. Measure its length. Then add the weights one by one, each time measuring the length of the spring as it stretches.

18. *Fall:* The time it takes for Lab Gear blocks to fall as a function of the number of blocks — **You will need** a stopwatch and 20 or more x^2-blocks. Line up x^2-blocks so that if the first one is pushed, each block will knock down the next block in succession.

19. *Summer:* The time it takes to do "the wave" as a function of the number of people involved. **You will need** a stopwatch. Decide on an order for the wave. Appoint a student (or the teacher) to be the timer. When the timer says "Go," take turns getting up and sitting down. Repeat the experiment for different numbers of people.

20. *Winter:* The height of an ice column as a function of the height of the corresponding water column — **You will need** some drinking straws, chewing gum, ice (or access to a refrigerator). Plug the bottom of a straw with gum. Fill it to a certain height with water. Mark and measure the height of the water column. Do it again with different amounts of water in other straws. Freeze them. Mark and measure the height of the column of ice.

21. Report Write an illustrated lab report on an experiment you conducted. This can be one of the ones presented in this section or another one of your own design. Include the data you collected, a graph, and an equation, if you found one. Describe the conditions in which you conducted the experiment, your expectations, and your conclusions.

THINKING WRITING 12.A Equations from Data

Each of the tables below gives four (x, y) pairs for a function. Each function is one of the following types and has an equation of the corresponding form.

Type of Function	Form of Equation
direct variation	$y = mx$
inverse variation	$y = k/x$
linear	$y = mx + b$

For each table in problems 1 through 6,

a. decide whether the function is direct variation, inverse variation, or linear;

b. find the equation of the function.

1.

x	y
0.05	5
0.5	0.5
5	0.05
50	0.005

2.

x	y
0.05	0.002
0.5	0.02
5	0.2
50	2

3.

x	y
0.9	0.6
1.5	1.0
2.7	1.8
5.1	3.4

4.

x	y
200	125
100	62.5
120	75
320	200

5.

x	y
0.01	0.73
0.1	0.55
1.5	-2.25
3	-5.25

6.

x	y
4	-2
8	-1
18	1.5
25	3.25

7. Each of the following three tests can be used to recognize a certain type of function among direct variations, inverse variations, and linear functions. Match the test to the type of function. Make sure your answer works for problems 1-6.

a. constant xy product

b. constant slope

c. constant y/x ratio

Because of measurement error, the numbers obtained in scientific experiments do not usually give perfect number patterns. For tables 8-10, find an equation that is approximately right.

8.

x	y
1.5	0.50
1.6	0.53
1.7	0.55
1.8	0.60
1.9	0.63
2.0	0.65

9.

x	y
12.5	6.8
13	6.5
13.5	6.3
14	6.1
14.5	5.9
15	5.6

10.

x	y
0.6	4.12
0.7	4.26
0.8	4.37
0.9	4.49
1.0	4.61
1.1	4.71

11. Report Summarize what you know about how to find the equation corresponding to experimental data, if it is one of the following types:

- direct variation
- linear function
- inverse variation

Include examples. Explain both how to recognize the type of function and how to find the actual equation.

LESSON 12.5

Modeling Motion

You will need:

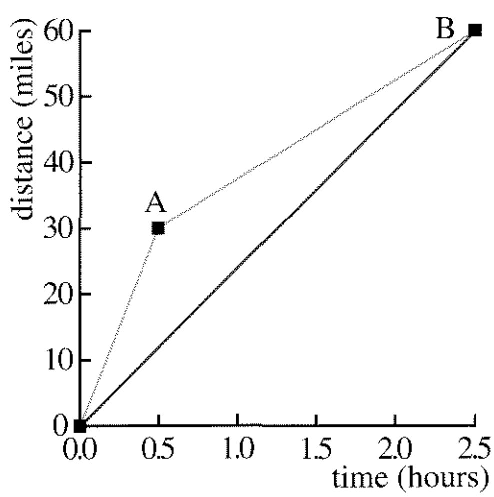
graph paper

AVERAGE SPEED

Definition: Average speed is total distance traveled divided by total travel time.

1. Joan goes to work at 6 A.M. She averages 60 mph on the interstate highway. She returns during rush hour, when she averages 15 mph. What is her average speed for the round trip if she travels 30 miles in each direction?

Many problems in this lesson can be understood better by making distance-time graphs, like this one about Joan's commute.



2. Explain how the coordinates of points A and B were obtained from the given information.

3. How does Joan's morning speed of 60 mph show up on the graph? Her return speed? Her average speed?

4. Joan calculated her average speed by adding the two speeds and dividing by 2.

$$(60 + 15)/2 = 37.5$$

Explain why this is wrong.

5. Jill traveled for two hours at 30 mph and two hours at 60 mph. Jack traveled for 90 miles at 30 mph and for 90 miles at 60 mph. Which of them had an average speed of 45 mph? Which one did not? Explain.

6. Generalization

a. I travel for t hours at v mph and t more hours at w mph. What is my average speed?

b. I travel to work, which is d miles away, at v mph, and travel back at w mph. What is my average speed?

RELAY RACE

Alaberg High's Track Team has a relay race team. These tables show the times in seconds of the individual runners in the 4 × 100 meter race at the meet with the Lean County School. The runners are listed in running order.

Alaberg	
Mal	12.2
Cal	12.0
Hal	12.4
Zal	11.4

Lean	
Neil	12.1
Neal	12.3
Alan	12.2
Allen	10.9

7. Imagine you are the radio announcer for this event. Describe the teams' performances. Who was ahead at various times? How did it end up? What was the key to the winning team's victory?

8. a. Compare the median running times for the two teams.
   b. Compare the mean running times for the two teams.
   c. Which is more relevant to winning the race?

9. Find each runner's speed in m/sec.

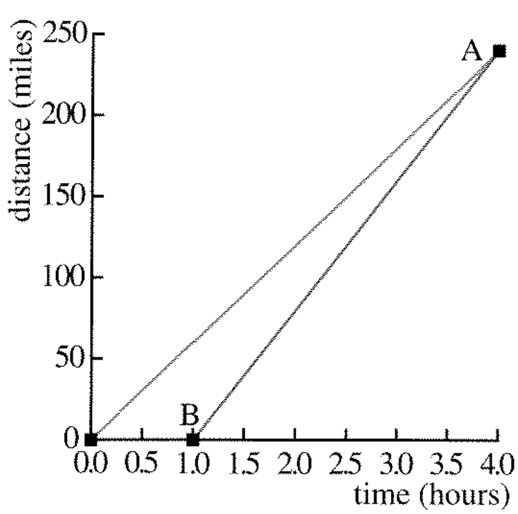
10. Find the average speed of each team in m/sec.

11. Show how each student answered problem 10 and find their answers.
    a. Andrea divided 100 by the mean running time for each team.
    b. Beth divided 400 by the total time for each team.
    c. Carolyn took the average of the individual members' speeds.

12. **Summary** Discuss the three methods presented in problem 11. Which ones are equivalent to each other? Which one is incorrect? Explain.

CATCH UP AND MEET

13. Jane is traveling at 60 mph along a road. She has traveled for four hours when Joe catches up with her. How fast must Joe have been traveling if he left the same place one hour after Jane?

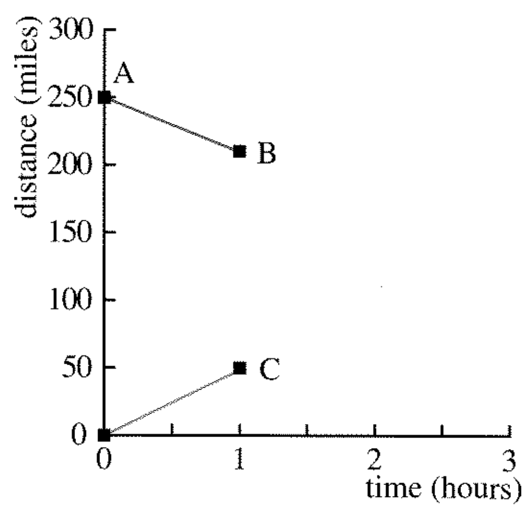


14. Look at the graph in problem 13. Explain how the coordinates of points A and B were chosen, and how the graph can help solve the problem.

15. Jim is traveling at 40 mph. Jorge leaves two hours later and travels at 50 miles per hour. How long until he catches up? How far have they gone?

16. Juan leaves at noon and travels at 45 mph. Jo leaves two hours later. How fast must she travel to catch up by:
    a. 8 P.M.? b. 8:30 P.M.?
    c. 11 P.M.? d. H P.M.?

17. Jacquey and Gigi start out at the same time, traveling towards each other. Jacquey travels at 50 mph. Gigi travels at 40 mph. They start out 250 miles apart. When and where do they meet?



18. The graph shows Jacquey and Gigi's progress during the first hour. (Jacquey's graph starts at the origin.) Explain how the coordinates of points A, B, and C were obtained, and how to use a graph like this to solve problem 17.

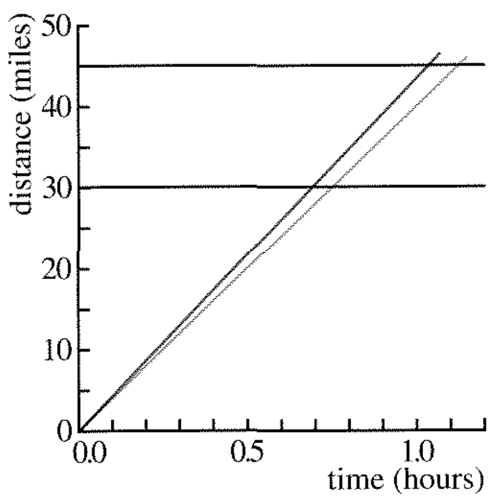
19. Greg starts out going towards Cary, traveling 50 mph. Cary starts out two hours later going 40 mph, going towards Greg. If they are 250 miles apart to begin with, when and where do they meet?

SAVING TIME

Paige travels to work so early that he meets hardly any traffic. He can drive at the speed limit the whole way. He wishes that the speed limit, which is 40 mph, would be raised so that he could sleep a little later in the morning.

20. How many minutes would Paige save if the speed limit were raised to 45 mph and he lives 30 miles from work?

21. Tara lives on the same road, 45 miles from work. How much time would she save?



22. Explain how you can use a graph like this one to think about problems 20 and 21.

23. Generalization How much time would be saved for people who live d miles from work if the speed limit were raised from 40 to 45 mph?

24. If Leon lives 60 miles from work, to what would the speed limit have to be raised (from 40 mph) in order for him to save

a. 6 minutes? b. 12 minutes?

25. Rina is taking a 60-mile trip. Which is greater: the time saved if she can travel 50 mph instead of 40 mph, OR the time saved if she can travel 60 mph instead of 50 mph? Explain.

LESSON 12.6

Gearing Up

In this lesson you will learn about the mathematics of gears. This will help you understand the decisions people have to make when they buy or design bicycles.

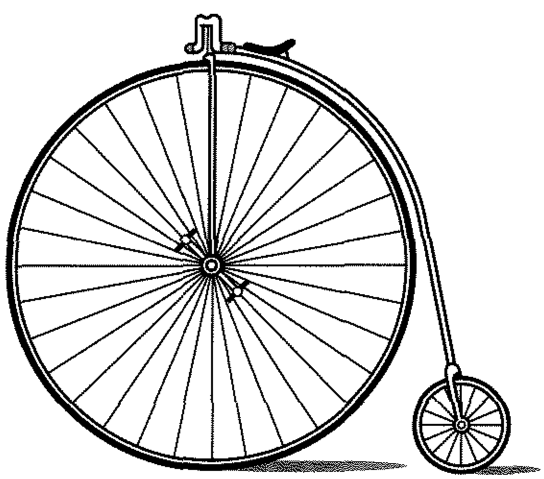
BIG WHEELS

1. How far does a bicycle travel for every revolution of the wheel for each wheel diameter below?

a. 20 in. b. 27 in.

c. 50 in. d. 64 in.

Old-fashioned bicycles had huge front wheels. Most of these high-wheelers, as they were called, had a 50-inch front wheel and a 17-inch rear wheel, but some of the makers got carried away and built front wheels as high 64 inches! The pedals were in the center of the front wheels.



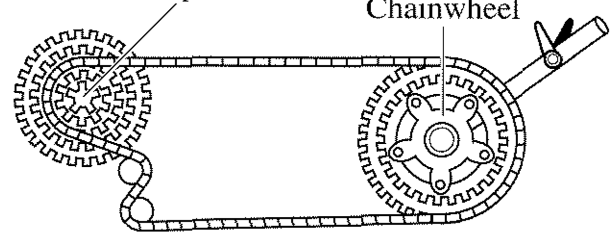
2. Why did bicycle makers make such big wheels?

Highwheelers had two drawbacks. First, the rider had to work very hard to get started, and most of these bicycles had to be pushed or dragged up hills. Second, their height made them a dangerous and impractical means of transportation. The rider had to jump down from the seat when the bicycle stopped, hoping to land feet-first.

The invention of gears on bicycles was a key development. Gears allowed the rider to travel longer distances for each turn of the pedals, without requiring such big wheels.

HOW GEARS WORK

Example: A bicycle has a chainwheel having 45 teeth and a rear sprocket having 15 teeth.



Each time the chain passes over one tooth on the chainwheel, it also passes over one tooth on the rear sprocket. Therefore, the rear sprocket will go through three revolutions for every one revolution of the chainwheel.

3. Explain why riding a 27-inch bicycle having these gears would be like riding an 81-inch bicycle in terms of the distance covered in one turn of the pedals.

Definition: The *gear ratio* is the ratio of the number of teeth on the chainwheel to the number of teeth on the rear sprocket.

4. If the gear ratio is 2.5, how many turns does the rear wheel make for each turn of the pedals?

A ten-speed bicycle has two chainwheels and five rear sprockets. Each combination of chainwheel and sprocket is a different gear.

5. Make a table to show how the gear ratio changes as a function of the number of teeth on the gears of a ten-speed bicycle, with two chainwheels having 40 and 54 teeth, and five rear sprockets having 14, 17, 22, 28, and 34 teeth.

	14	17	22	28	34
40	—	—	—	—	—
54	—	—	—	—	—

6. Generalization Write the gear ratio (g) as a function of the number of teeth on the chainwheel (c) and the number of teeth on the rear sprocket (r).

GEAR

Definition: The *gear* is the gear ratio multiplied by the diameter of the rear wheel. It gives the diameter of the wheel that would travel the same distance in one revolution of the pedals. (The unit of gear is inches, but it is usually omitted.)

Example: The gear ratio is 40/20, or 2, when using a chainwheel having 40 teeth and a rear sprocket having 20 teeth. On a bicycle having 26-inch wheels, the gear would be 2×26, or 52. This means that each turn of the pedals when the bicycle is in this gear would move the bike a distance equivalent to one turn of a 52-inch wheel.

7. If the gear is 52, how far would the bike travel with each turn of the pedals?

8. Generalization

a. Write the gear (G) as a function of the number of teeth on the chainwheel (c), the number of teeth on the rear sprocket (r), and the size of the wheel (w).

b. If the gear is G, how far would the bike travel with every turn of the pedals?

c. Write a formula that gives the distance (d) that the bike would travel with each turn of the pedals as a function of c, r, and w.

CADENCE

Definition: The *cadence* is the pace of pedaling.

A good cadence to maintain is 65 to 85 pedal revolutions per minute. Better cyclists like to maintain a cadence of 90 turns per minute.

9. Julio's ten-speed bike has wheels 27 inches in diameter. Its gears were described in problem 5. At a cadence of 90 pedal revolutions per minute, how fast, in miles per hour, would Julio be going in the highest gear? (Hint: Find a conversion factor to get directly from pedal revolutions per minute to miles per hour.)

10. If Julio knows his cadence, find a way for him to calculate his speed mentally in miles per hour when riding in the highest gear.

DESIGN A BIKE

11. Project Design a bicycle. First describe the future owner of the bicycle and his or her needs. Will the rider be climbing steep hills? Be racing? Choose a size for the wheel, and the number of teeth for the gears of a 10-, 15-, or 18-speed bicycle. The following information may be helpful. Describe how each gear would most likely be used.

Wheel diameters

24, 26, and 27 inches are common.

Teeth on the chainwheel

24 to 58

Teeth on the rear sprocket

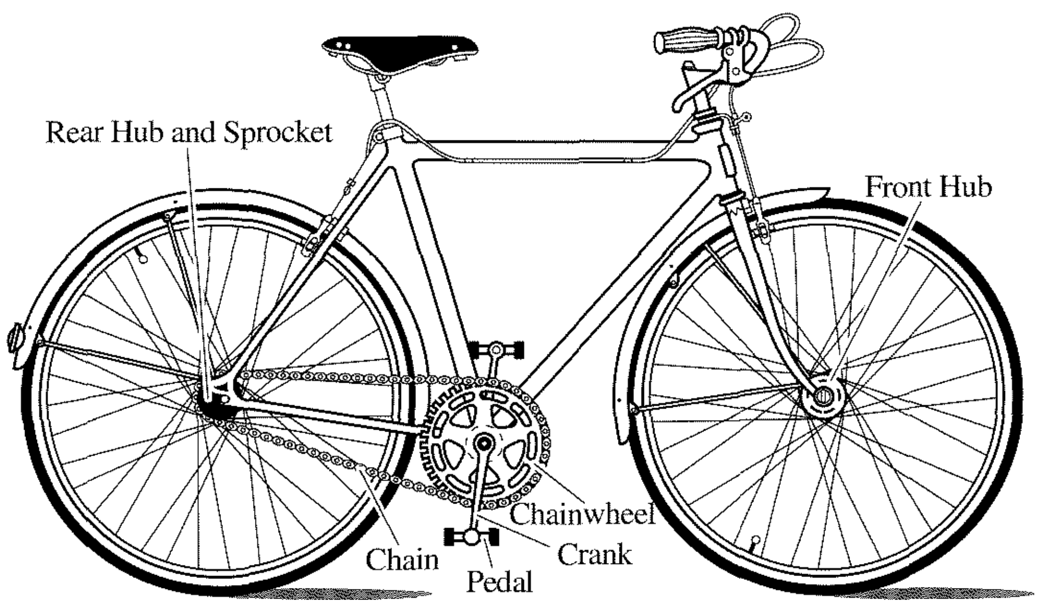
12 to 38

Sample Gears

- Very low gear, for climbing steep hills and for easy starts: 33
- Medium gear, for general use: 54
- Very high gear, for going downhill fast, and for racing: 100

Progression

Some cyclists like an approximately geometric progression of gears, because the common ratio makes the change feel the same from one gear to the next.



LESSON 12.7

Iterating Linear Functions

You will need:

graph paper

or function diagram paper

TREE HARVESTING

Paul's Forestry Products owns two stands of trees. This year there are about 4500 trees in Lean County and 5500 in Cool County. So as not to run out of trees, the yearly harvesting policy at each location is to cut down 30% of the trees and then plant 1600 trees. For example, in Lean County this year they will cut 1350 trees and plant 1600 trees.

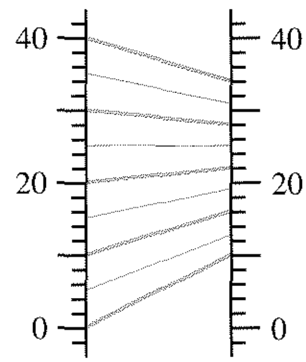
1. Make a table of values showing how many trees they would have at each location every year for nine years.
2. Describe the change in the number of trees at each location. Is it increasing or decreasing? Is it changing at a constant rate from year to year? What do you think will happen in the long run?
3. Write a formula that would give the number of trees next year in terms of the number of trees this year. (Use y for next year's number and x for this year's number. What you get is called a *recurrence equation.*)
4. How many trees would they have at each location after 30 years?

DRUGS

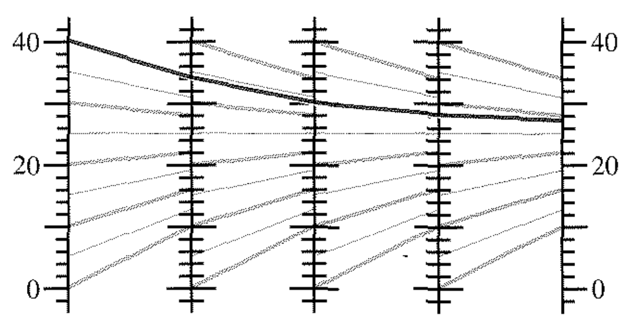
To control a medical condition, Shine takes ten milligrams of a certain drug once a day. Her body gets rid of 40% of the drug in a 24-hour period. To find out how much of the drug she ends up with over the long run, we can use function diagrams.

5. If x is the amount of the drug Shine takes per day, and y is the amount that ends up in her body over the long run, explain why the recurrence equation is $y = 0.6x + 10$.
6. Make a table of values for the recurrence equation, using these values for x, the daily dose: 0, 5, 10, 15, 20, 25, 30, 35, 40.

Here is a function diagram for the recurrence equation.



This function diagram can be repeated to show what happens over the long run. The linked diagrams show how the y-values for one become the x-values for the next.



7. Use the diagram to predict what happens in the long run if Shine takes 10 mg a day of the drug after an initial dose of:
   a. 10 mg; b. 25 mg;
   c. 40 mg.
8. Check your predictions by calculation.

Remember that instead of linked diagrams like in the figure, you could use a single function diagram of the function. Just follow an in-out line, then move horizontally across back to the x-number line; then repeat the process, using the in-out line that starts at that point.

SAVINGS

Glinda puts $50 a month into a savings account paying yearly compound interest of 6%.

9. What is the interest per month?

10. How much money will she have at the end of one year?

11. Write a recurrence equation for problem 10, expressing the amount in the account at the end of each month as a function of the amount the previous month.

12. Make a function diagram.

13. How does what happens in the long run for problem 10 differ from the problems in the previous sections? Explain.

Definition: To *iterate* a function means to use its output as a new input.

All the problems in this lesson involve iterating linear functions. We will use function diagrams and algebraic symbols to get a more general understanding of this kind of problem.

14. Describe the difference between function diagrams for $y = mx + b$ for the following:

a. $0 < m < 1$ b. $m = 1$
c. m > 1

THE FIXED POINT

Definition: A fixed point of a function is one in which the output is the same as the input.

Example: For the function $y = 7x - 12$, when the input is 2, the output is also 2.

15. What is the fixed point for each of the functions in problems 3 and 5? Why was it important in understanding the problems?

16. Find the fixed points.

a. $y = 3x - 6$ b. $y = 3x + 5$
c. $y = 3x$ d. $y = x$
e. $y = x + 3$ f. $y = x^2 - 2$

17. Function diagrams may help you think about these questions.

a. There is a linear function that has more than one fixed point. What is it? Explain.
b. What linear functions have no fixed points? Explain.

18. Generalization

a. Find a formula for the fixed point for the function $y = mx + b$. (Hint: Since the output is the same as the input, substitute x for y and solve for x.)
b. Explain why $m = 1$ is not acceptable in the formula you found. What does that mean in terms of the existence of the fixed point for equations of the form $y = x + b$?

ANALYZING THE SEQUENCES

When iterating a function, you get a sequence of numbers.

19. Exploration Start with the equation $y = 2x + 3$. Change one number in the equation so that when iterating the function, starting with any input, you get

a. an arithmetic sequence;
b. a geometric sequence;
c. a sequence where the values get closer and closer to a fixed point.

Compare your answers with other students' answers.

20. Generalization When iterating $y = mx + b$, different things may happen, depending upon the value of the parameters. Find the values of m and b which lead to the following situations:

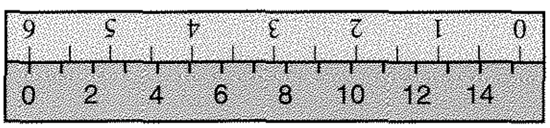
a. arithmetic sequences;
b. geometric sequences;
c. sequences where the values get farther and farther from the fixed point;
d. sequences where the values get closer and closer to the fixed point.

21. Report Summarize what you know about iterating linear functions. Include, but do not limit yourself to these topics.

- real-world applications
- use of function diagrams
- the fixed point
- these special cases:
  - ◊ $b = 0$
  - ◊ $0 < m < 1$
  - ◊ $m = 1$
  - ◊ $m > 1$

DISCOVERY ***TWO RULERS***

Alice's ruler



Oliver's ruler

Alice and Oliver lined up her inch ruler against his centimeter ruler, as in the above figure. This yielded the following table of numbers.

x	y
0	6.0
2	5.2
4	4.4
6	3.6

22. a. Graph these data.
b. What is the equation for y in terms of x?
c. Interpret the slope and y-intercept in terms of the rulers.

Place an inch ruler and a centimeter ruler against each other so that they run in opposite directions.

23. Using the ruler arrangement you made as a source of (x, y) pairs, make a table like Alice's and Oliver's. Then make a graph.

24. Write an equation for the function that shows the relationship between the numbers in your table.

25. Interpret the slope and y-intercept in your equation and graph in terms of your rulers and their positions.

LESSON 12.8 Representing Functions

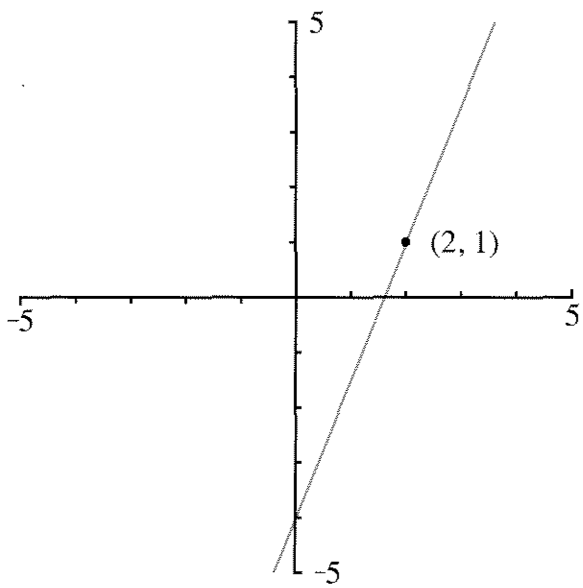
You will need:

- graph paper
- function diagram paper
- colored pens

POINTS AND LINES

As you know, an (x, y) pair is represented as a point on a Cartesian graph and as an in-out line on a function diagram. In this section we will review how an equation of the form $y = mx + b$ is represented in these formats.

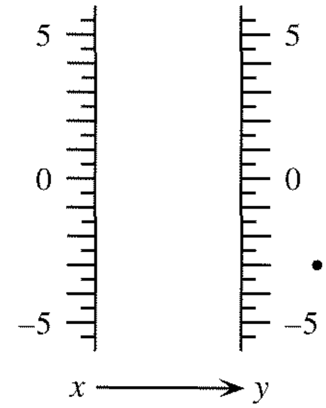
1. For the function represented by this Cartesian graph,
   a. write the equation;
   b. draw a function diagram.



2. Extend the in-out lines in the function diagram you made in problem 1. They should meet in one point, called the focus.

3. What is the minimum number of lines you need to draw to find the focus? Explain.

Actually, a function of the form $y = mx + b$ can be represented by just the focus, as you will see in the next problem.



4. The figure shows the focus of a certain function of the form $y = mx + b$.
   a. Place a ruler on the focus, and find three in-out lines. Do not draw the lines, but keep a record of the (x, y) pairs.
   b. Find the equation.

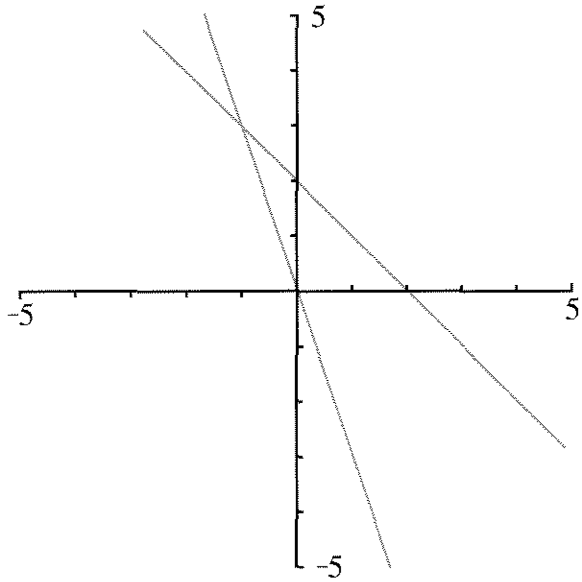
5. If you were to make a Cartesian graph of this function, what is the minimum number of points you would need to plot? Explain.

This table shows how points and lines appear in the two representations. Notice how points and lines are switched when going from one representation to the other.

Object	Representation	
	on Cartesian graph	on function diagram
(x, y) pair	one point	one line (in-out)
linear equation	one line	one point (the focus)

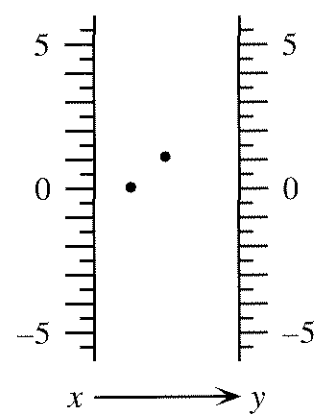
FOCUS ON SIMULTANEOUS EQUATIONS

This graph shows $y = -3x$ and $y = -x + 2$.



6. What point do the two lines have in common?

The function diagram shows the foci of $y = -3x$ and $y = -x + 2$. (Foci is the plural of focus.)



7. Check that the foci are placed correctly.

a. Place a ruler on the focus and find three in-out lines for each function. Do not draw the lines, but keep a record of the (x, y) pairs.

b. Check that the (x, y) pairs you found satisfy the equations.

8. If you were to draw an in-out line containing both foci, what (x, y) pair would it represent?

9. How is the solution of a system of linear equations represented on:

a. a Cartesian graph?

b. a function diagram?

10. Explain how to solve this system by using a function diagram. (Hint: First find each focus, then find the solutions.)

$$\begin{cases} y = 0.5x + 2 \\ y = 2x - 1 \end{cases}$$

Because two lines meet in a point, the solution to a system of simultaneous equations is represented on a Cartesian graph by a point. Because two points determine a line, the solution to a system of linear equations is represented on a function diagram by a line (an in-out line).

FAMILIES OF FUNCTIONS

Definition: A *family* of functions is a group of functions that share a certain attribute.

11. All functions having equations of the form $y = 5x + b$ belong to the $m = 5$ family.

a. Sketch the graphs of two members of the family.

b. What do all graphs for this family have in common?

12. All functions having equations of the form $y = mx + 7$ belong to the $b = 7$ family.

a. Sketch the graphs of two members of the family.

b. What do all graphs for this family have in common?

All functions in the same b-family have foci that lie on the same in-out line.

13. These four functions are in the same b-family. For each one, draw in-out lines to find the focus and mark it with a colored pen or pencil. Do all four on the same diagram.

a. $y = 0.5x - 2$ b. $y = 2x - 2$
c. $y = -2x - 2$ d. $y = -0.5x - 2$

14. What is the family name for the functions in problem 13?

15. Why do all the foci of the functions in problem 13 lie on the same in-out line? Which in-out line is it? Explain.

16. The foci for all functions in the family $b = -3$ also lie on one in-out line. Which line? Explain how you know.

Many m-families also have foci that lie on the same in-out line in a function diagram.

17. a. On a function diagram, find and mark the focus for $y = -2x + 3$.
b. On the same function diagram, find and mark the focus for $y = -2x + 1$.
c. Find and mark the focus for several other graphs of the form $y = -2x + k$.

18. What is the family name for all the functions in problem 17? Explain why the foci are all on the same line. Describe the line.

19. If two functions both have a focus on the same vertical line, what would their Cartesian graphs have in common?

20. What is the family name for all functions having focus half-way between the two number lines?

21. There is one m family for which the function diagrams have no focus, because the in-out lines do not meet. Which m family is this?

22. Summary On a function diagram, what is true of the foci of all linear functions in the same

a. m family? b. b family?

23. The functions representing Charles's Law for gases in the graph in Lesson 4 form a family that is neither an m nor a b family. If you were to make function diagrams for them, the foci would all be on a certain in-out line. Which one? Explain.

THINKING WRITING

12.B V-Shaped Graphs

You will need:

- graph paper
- graphing calculator (optional)

THE SQUARE ROOT OF x^2

As you know, the radical sign means *the non-negative square root of.*

1. Make a table of values, and a graph, for the function $y = \sqrt{x^2}$. Use at least six values for x, including positive numbers, negative numbers, and zero.

2. Find a linear function that has the same graph as $y = \sqrt{x^2}$, when
a. x is positive; b. x is negative.

3. True or False? $\sqrt{x^2} = x$. Explain.

ABSOLUTE VALUE

As you may remember, the absolute value of a number is the distance between that number and zero.

4. Repeat problems 1-3 for the function $y = |x|$.

Graph the functions in problems 5 through 10. Use separate axes for each one. Write each equation on its graph.

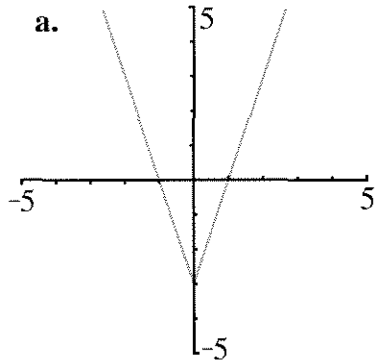
5. $y = |x| + 2$ **6.** $y = |x| - 2$

7. $y = -|x|$ **8.** $y = 2|x|$

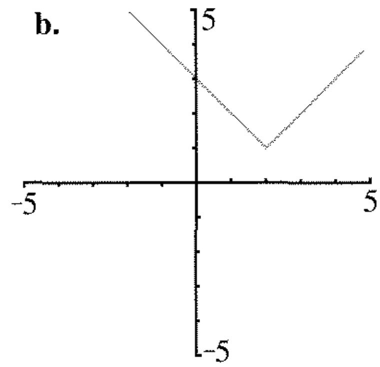
9. $y = |x + 2|$ **10.** $y = |x - 2|$

11. **Exploration** Find equations of the form $y = A|x - H| + V$ for these four graphs.

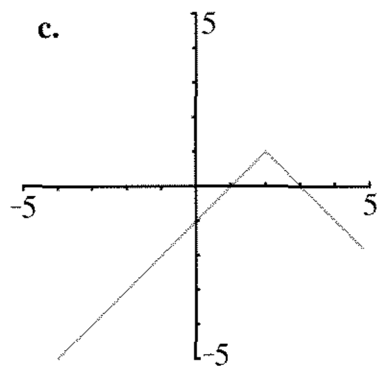
a.



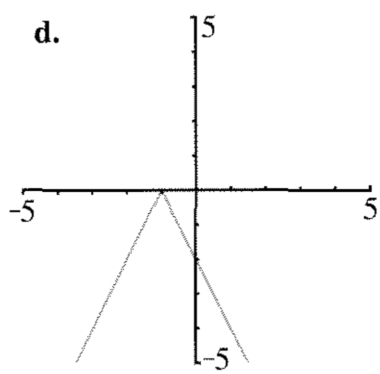
b.



c.



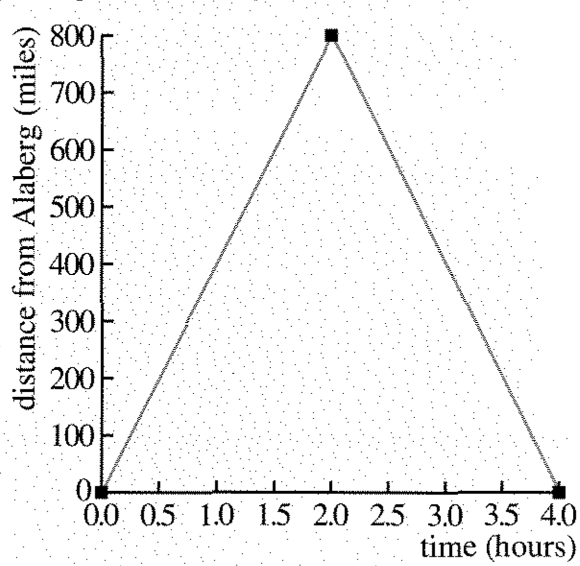
d.



12. **Report** Write an illustrated report describing graphs of the form $y = A|x - H| + V$. Describe how each of the parameters A, H, and V affects the graph. What are the slopes? Where is the vertex? What are the domain and range? Give examples, including both negative and positive values of all the parameters.

A ROUND TRIP

This graph shows a plane's trip. It was sighted passing over Alaberg at time $t = 0$.



13. Describe the plane's trip.

14. The equation of the graph is of the form $y = A|x - H| + V$. What are A, H, and V?

15. If the plane were going at 300 miles per hour,
    a. how would the graph be different?
    b. how would the equation be different?

REVIEW LIKE TERMS

When combining terms involving fractions, it is sometimes useful to write the fractions with common denominators. However, it is often more convenient to use the method that is demonstrated in the following example.

Example: $\frac{x}{60} - \frac{11x}{70}$

$$= \frac{1}{60}x - \frac{11}{70}x$$

$$= \left(\frac{1}{60} - \frac{11}{70}\right)x$$

$$= -0.14x$$

(A calculator was used for the last step.)

Combine like terms.

16. $\frac{2x}{3} - 4x$

17. $\frac{5x}{6} + \frac{7}{8} + \frac{9x}{4}$

18. $\frac{3x + 2}{5} - \frac{x}{2}$

Essential Ideas

FIRST CLASS STAMPS

This table shows the costs in cents of first-class stamps over the past sixty years. The dates indicate the year when there was an increase in the first-class rate.

Year	Cost
1932	3
1958	4
1963	5
1968	6
1971	8
1974	10

Year	Cost
1975	13
1978	18
1981	20
1985	22
1988	25
1991	29

Interpolation is not relevant since all the data are known within the given period. However extrapolation may be possible.

1. Graph the data as a step function. For example, the cost was 3 cents from 1932 to 1957.
2. In 1985 Barbara wanted to use the average cost increase in the period 1932-1985 to predict the cost of stamps in 1991.
   a. What was the average yearly increase?
   b. Based on this, what cost did she predict for 1991?
3. In 1985 Sue used a computer to find the average percent increase over the 53-year period. The computer indicated that on the average, the cost went up by 3.8% a year. Based on this, what cost did she predict for 1991?

In 1991 they used the same methods to find the average increases over the 59-year period. Barbara found an average increase of 0.44 cents a year, and Sue found an average percent increase of 3.9% a year.

4. Make a prediction for the cost of stamps in the year 1999 and 2032. Explain.

THE MILE RUN

Year	Time
1868	4:29.0
1868	4:28.8
1874	4:26.0
1875	4:24.5
1880	4:23.2
1882	4:21.4
1882	4:19.4
1884	4:18.4
1894	4:18.2
1895	4:17.0
1911	4:15.6
1911	4:15.4
1913	4:14.6
1915	4:12.6
1923	4:10.4
1931	4:09.2
1933	4:07.6
1934	4:06.8
1937	4:06.4
1942	4:06.2

Year	Time
1942	4:04.6
1943	4:02.6
1944	4:01.6
1945	4:01.4
1954	3:59.4
1954	3:58.0
1957	3:57.2
1958	3:54.5
1962	3:54.4
1964	3:54.1
1965	3:53.6
1966	3:51.3
1967	3:51.1
1975	3:51.0
1975	3:49.4
1979	3:49.0
1980	3:48.8
1981	3:48.53
1981	3:48.40
1981	3:47.33

5. The table shows the world record for the mile run from 1868 to 1981. Plot the time *in seconds* as a function of year.

6. Use the median-median line method to fit a line.

7. What is the equation of your fitted line?

8. Richard Webster of Great Britain ran the mile in 4:36.5 in 1865. How does this compare with the time for 1865 predicted by your fitted line?

9. Steve Cram of Great Britain ran one mile in 3:46.31 in 1985. How does this compare with the time predicted by your fitted line?

10. a. According to your model, when would the mile be run in 0 seconds?
    b. For how many more years do you think your fitted line will be a good predictor of the time?

WIRES

When a metal wire changes temperature, it expands or contracts, according to the equation

$$L = L_0(1 + kT),$$

where L is the length of the wire, L_0 is its length at 0°C, T is the temperature, and k depends on the metal. For copper, $k = 1.8(10^{-5})$.

11. A copper wire is 100.05 meters long at 40°C. If it is cooled to -10°C, how much will it shrink? (Hint: First find its length at 0°C.)

12. Two poles are 100 meters apart. They are connected by a 100.05-meter copper wire in the summer, when the temperature is 40°C. In the winter the temperature drops to -10°C.
    a. Explain why the wire breaks.
    b. How long should the wire be so as not to break in the winter?

A nickel-iron alloy is created. Measurements are made in a lab on a wire made of the alloy. It is found that a wire that is 10 meters long at 0°C expands by one half a millimeter at 100°C. The alloy is called *Invar.*

13. Find the value of k for Invar.

14. Would an Invar wire that measures 100.01 meters at 40° C work to connect the poles in problem 12? Explain.

EQUATIONS FROM DATA

a.

x	y
0.4	15
0.6	10
0.8	7.5
1	6

b.

x	y
0.4	0.667
0.6	1.00
0.8	1.33
1	1.67

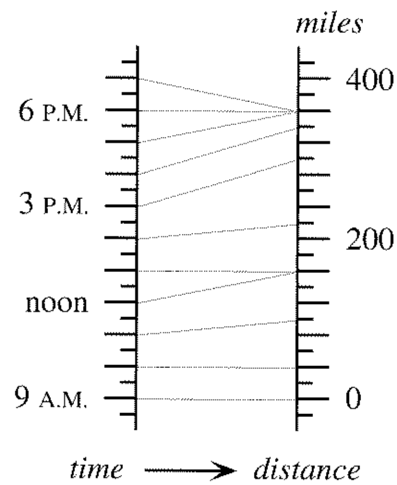
c.

x	y
0.4	-4.4
0.6	-2.6
0.8	-0.8
1	1

15. Find an equation for each table. Hint: One is a direct variation, one an inverse variation, and one a linear function.

THE CAR TRIP AND THE BICYCLE TRIP

Reread **Thinking/Writing 2.B** (Chapter 2). The function diagram is shown below.

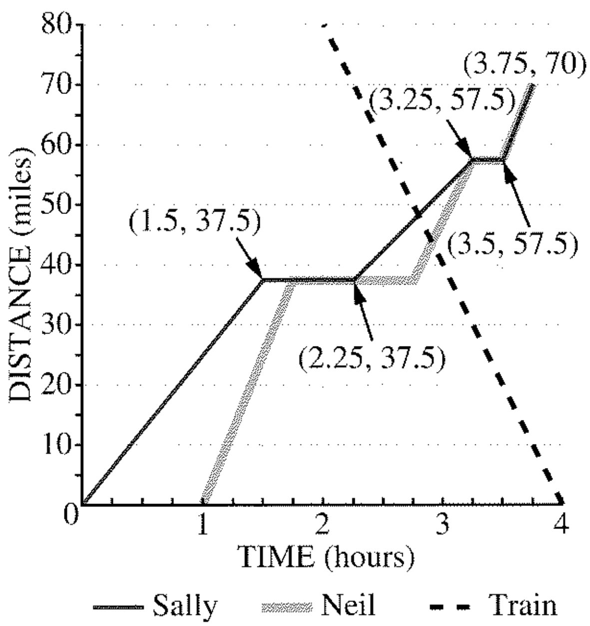


Car Trip

16. Make a Cartesian graph for the car trip, as best you can from the information given.

17. What is the car's average speed,
    a. if you include the time the car was stopped in the middle of the day?
    b. if you include only the driving time?

Reread **Thinking/Writing 4.A** (Chapter 4). The graph is shown below.



18. Repeat problem 17 for the van Neil drove.

19. If Neil were to make the return trip in the same length of time, but traveling at a constant speed and never stopping, what would be his speed?

20. a. Write an equation for Sally's graph during the leg of the trip when she and the train passed each other.
    b. Solve the system of equations consisting of the equations representing Sally's and the train's motion.
    c. Interpret the point of intersection.

ASTHMA

For her asthma Lynne takes 360 mg of the drug *theophylline* twice a day. After 12 hours, 60% of the drug has been eliminated from her body.

21. Assume Lynne has x_a mg of the drug in her body immediately after taking the dose. Explain why $y_a = 0.4y_a = 0.4x_a + 360$ is the recurrence equation that says how much will be in her body immediately after taking the next dose.

22. Assume she has x_b mg of the drug in her body immediately before taking the dose. Explain why $y_b = 0.4(x_b + 360)$ is the recurrence equation that says how much will be in her body immediately before taking the next dose.

The amount of theophylline in Lynne's body is constantly changing, but the lowest amount (right before taking the drug) and the highest amount (right after) eventually approach a stable level.

23. Find that level, using tables, function diagrams, or equations. What is the level before taking the dose? What is it after?

CHAPTER 13

The spiraling vortex of a whirlpool

Coming in this chapter:

Exploration You want to make pens for Stripe, your pet zebra, and Polka Dot, your pet leopard. You have 100 feet of fencing. If you use all of it to make two pens of equal area, what is the biggest area possible?