CHAPTER 8

The spiral surface pattern of gastropod fossils

Coming in this chapter:

Exploration A population is growing at a rate of about 2% per year. In how many years will the population double? Experiment with different starting values for the population. How does your answer depend on the starting value?

GROWTH AND CHANGE

LESSON 8.1

Height and Weight

You will need: graph paper

Dr. Terwit, a pediatrician, kept records of her son Joshua's height and weight from birth to age four years. We will use these numbers to learn about *rate of change.*

Age	Height (cm)	Weight (kg)
birth	51	3.4
3 mo	60	5.7
6 mo	66	7.6
9 mo	71	9.1
12 mo	75	10.1
15 mo	79	10.8
18 mo	82	11.4
2 yr	88	12.6
2.5 yr	92	13.6
3 yr	96	14.6
4 yr	103	16.5

HEIGHT AS A FUNCTION OF AGE

1. Make a graph to represent height as a function of age. (Note that the ages given are not evenly spaced.)

2. What is the increase in height between:
a. birth and three months?
b. 15 months and 18 months?
c. birth and one year?
d. three years and four years?

3. Did Joshua's height grow faster or more slowly as he grew older? Explain your answer by referring to:
a. the answers to problem 2;
b. the shape of the graph.

4. If Joshua had grown the same number of centimeters every month, what would his average rate of growth be, in *centimeters per month,* between:
a. birth and three months?
b. 15 months and 18 months?
c. birth and one year?
d. three years and four years?

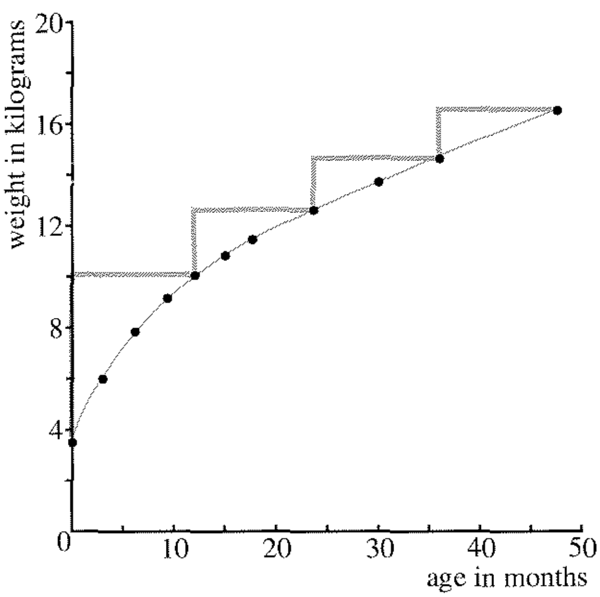
5. What was Joshua's *average rate of growth* in centimeters per month during his first four years? Compare this average with the averages you found in problem 4.

6. **Summary** Write a short paragraph summarizing the relationship between Joshua's age, his height, and the rate of his growth. In particular, explain the idea of average rate of growth and how it changed with his age.

7. **Project** Find out how many sizes there are for babies' and children's clothes in the age range studied here. Is what you find consistent with the information in the table?

WEIGHT AS A FUNCTION OF AGE

This is a graph of weight as a function of age. The straight lines form four *steps* connecting some data points.



8. Use the data to answer these questions about the graph.
   a. How high is each step? (Give your answer in kilograms.)
   b. How wide is each step? (Give your answer in months.)
   c. Explain the meaning of your answers to (a) and (b) in terms of the *yearly* change in Joshua's weight.

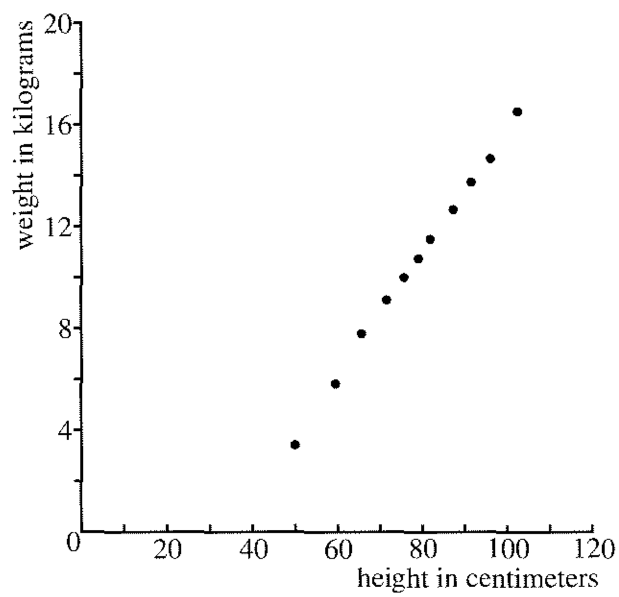
9. Find the average *monthly* weight gain between ages
   a. two and two-and-a-half;
   b. two-and-a-half and three;
   c. two and three.

10. Joshua's weight grew at a fairly constant monthly rate between ages one and four. Explain how this can be seen
    a. on the graph;
    b. numerically.

11. However, his weight grew much more slowly between ages one and four than during his first year. Explain how this can be seen
    a. on the graph;
    b. numerically.

WEIGHT AS A FUNCTION OF HEIGHT

This is a graph of weight as a function of height.



12. How much weight did Joshua gain for each centimeter he gained in height? Answer this question for the following periods:
    a. birth and three months;
    b. ages three and four;
    c. on the average, over the four years.

13. Study the preceding graph and table and make calculations to find the time in Joshua's first four years when he gained
a. the least weight per centimeter;
b. the most weight per centimeter.

14. Compare the two graphs of weight (as a function of age and as a function of height). How are they alike? How are they different? Discuss the shape of the graphs, the units, and the rate of change.

Because the rate of change of weight as a function of height does not vary much, the data points fall close to a line. You could say that this data is nearly linear. In cases like this, it is a common statistical technique to approximate the data with a line. You will learn more about this in future lessons, but first you need to know more about lines and linear functions.

BOYS AND GIRLS

The following table shows the average height in inches of boys and girls, ages 9 through 18.

	Height (in.)	
Age	**Girls**	**Boys**
9	52.3	53.3
10	54.6	55.2
11	57.0	56.8
12	59.8	58.9
13	61.8	61.0
14	62.8	64.0
15	63.4	66.1
16	63.9	67.8
17	64.0	68.4
18	64.0	68.7

15. Report Write a report comparing the height and the rate of growth of boys and girls. Include a graph showing the heights of both boys and girls as a function of age, on the same axes. (Since the graphs are close to each other, you may want to distinguish them by using color.) Your report should include, but not be limited to, answers to these questions.
- How many inches do boys and girls gain per year, on the average?
- At what ages do they grow fastest?
- How many inches do they gain per year during those growth spurts?

LESSON 8.2

Focus on Function Diagrams

You will need:
- graph paper

REVIEW PARALLEL-LINE DIAGRAMS

1. a. Draw a function diagram such that its in-out lines are *parallel* and going uphill (from left to right).
b. Find the function corresponding to the diagram, using an in-out table if you need it.

2. Repeat problem 1 with parallel in-out lines going
a. downhill;
b. horizontally.

3. For the functions you created in problems 1 and 2, when x increases by 1, by how much does y increase? Does it depend on the steepness of the lines?
(To answer this, compare your functions with other students' functions.) Explain your answer.

Problems 4 through 9 refer to the function diagrams shown on the next page.

THE FOCUS

Definition: If an in-out line is horizontal, its input is called a *fixed point.*

For example, both x and y equal 12 in diagram (a), so 12 is a fixed point for that function.

4. What are the fixed points for functions (b-p)?

Definition: In-out lines can be extended to the left or right. If all of them meet in a single point, that point is called the *focus.*

5. **Exploration** Consider the function diagrams shown in figures (a-p). For each one, find the function. You may split the work with other students. Describe any patterns you notice. If you cannot find all the functions or patterns, you will get another chance at the end of the lesson.

MAGNIFICATION

6. Look at function diagram (h). By how much does y change when x increases by:
a. 1? b. 2?
c. some amount A?

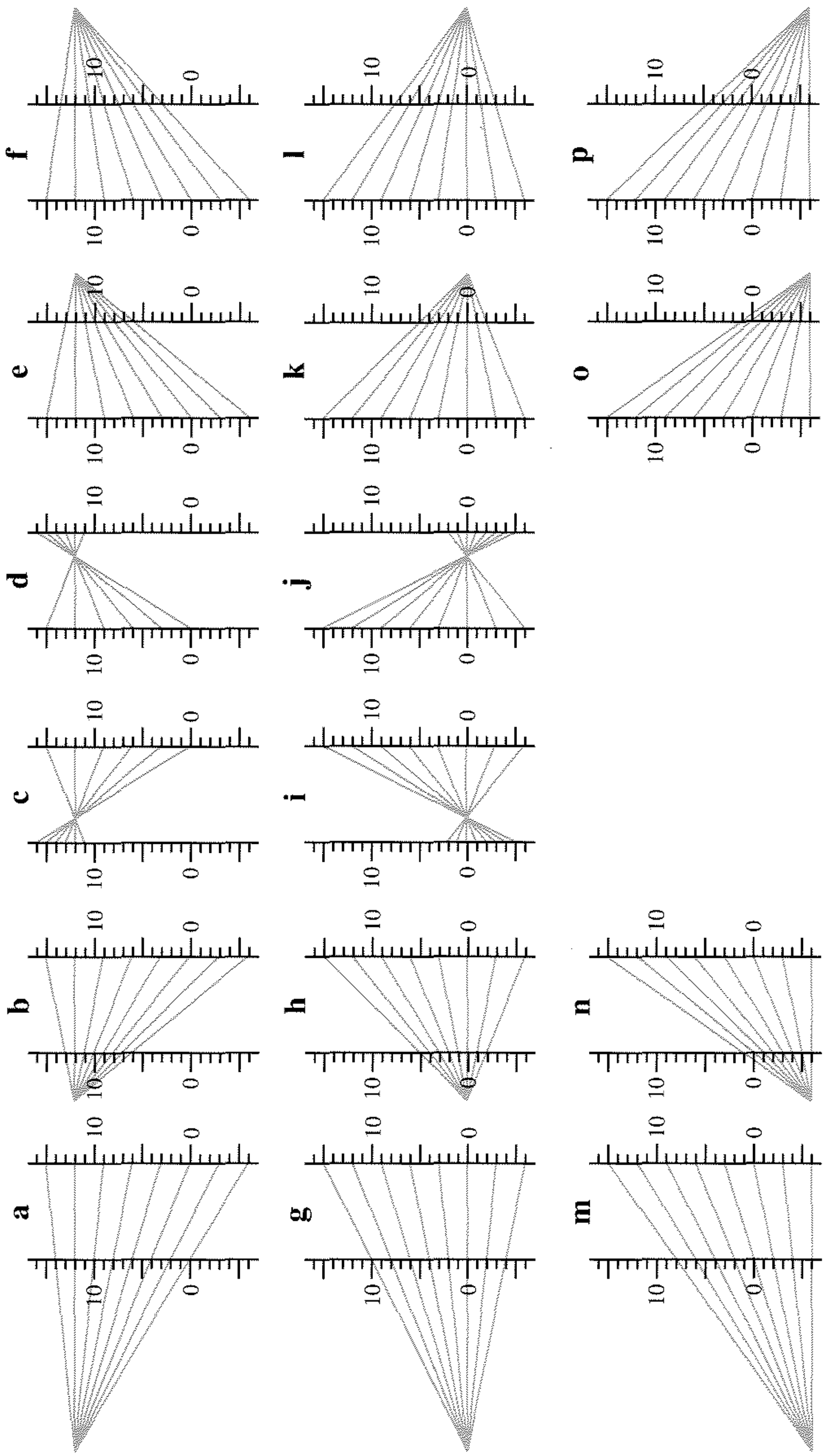
In function diagrams that have a focus, changes in y can be found by multiplying the changes in x by a certain number, called the *magnification.*
(change in x) · (magnification) = (change in y)

7. a. What is the magnification for (h)?
b. What other diagrams have the same magnification?

Rule: If y decreases when x increases, the magnification is negative.

8. For which diagrams is the magnification equal to -3? (If x increases by 1, y *decreases* by 3.)

9. Find the magnification for each function diagram. Note that the magnification can be positive or negative, a whole number or a fraction.



sixteen function diagrams

THE m PARAMETER

You probably noticed that all the function diagrams represent functions of the form $y = mx + b$. It turns out that this is always true of function diagrams with a focus. As you may remember, the letters m and b in the equation are called *parameters.*

10. Look at the equations you found in the Exploration, problem 5. What is the relationship between the magnification and the m parameter in those equations? Explain.

11. If you move the focus of a function diagram up, how does it affect the value of m? How about if you move it down?

12. Where would the focus be if m was
- a. a negative number?
- b. a number between 0 and 1?
- c. a number greater than 1?

13. What is a possible value of m if the focus is
- a. half-way between the x- and y-number lines?
- b. between the x- and y-number lines, but closer to x?
- c. between the x- and y-number lines, but closer to y?

14. What is a possible value of m if the focus is
- a. far to the left of the x-number line?
- b. close to the left of the x-number line?
- c. close to the right of the y-number line?
- d. far to the right of the y-number line?

15. In some parts of mathematics, parallel lines are said to meet at a point that is *at infinity.* In that sense, parallel-line diagrams could be said to have a focus at infinity. Is this consistent with your answer to problem 14? Explain.

RATE OF CHANGE

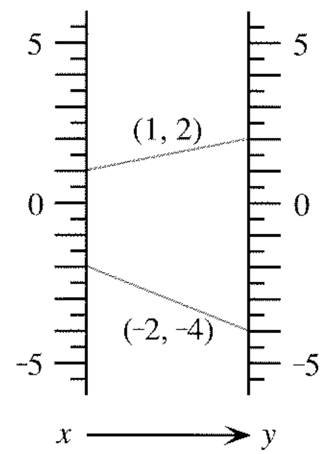
Once again, look at the diagrams (a-p).

16. On each diagram, *as x increases,* follow y with your finger. For what values of m does y
- a. go up?
- b. go down?
- c. move fast?
- d. move slowly?

The magnification is often called the *rate of change.*

17. What is the rate of change if y increases by 3 when x increases by:
- a. 1?
- b. 6?
- c. -10?

THE b PARAMETER



Two in-out lines are shown in the diagram. Each one is labeled with a number pair. The first number in the pair is the input, and the second number is the output.

Notation: Any in-out line can be identified by a number pair. From now on, we will refer to lines on function diagrams this way. For example, the line connecting 0 on the x-number line to 0 on the y-number line will be called the (0, 0) line.

18. What can you say about the b parameter if the focus is on the (0, 0) line?

19. Look at diagram (n). Its equation is $y = 3x + 12$.

a. Name the in-out lines that are shown.

b. Check that the pairs you listed actually satisfy the equation by substituting the input values for x.

c. Among the pairs you checked was (0, 12). Explain why using 0 as input gave the b parameter as output.

20. In most of the diagrams (a-p), there is an in-out line of the form (0, ___). How is the number in the blank related to the b parameter? Explain.

$y = mx + b$

21. If you did not find all the equations for the function diagrams (a-p), when working on problem 4, do it now. Hint: You may use what you learned about magnification and about the (0, ___) in-out line.

22. **Summary** Write what you learned about function diagrams, the fixed point, the focus, magnification, and the parameters m and b. Also mention parallel-line diagrams.

REVIEW *BINOMIAL MULTIPLICATION*

Multiply and combine like terms.

23. $(3x + 1)(x - 2)$

24. $(2x - 3)(5 - x)$

25. $(5 + x)(3x - 3)$

26. $(2y - 2)(6 - y)$

27. $(3x - 1)(2 + x)$

28. $(3x + 1)(2 + x)$

29. $(6 + y)(2y + 4)$

30. $(y - 4)(2y + 2)$

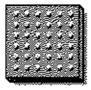
31. $(y - 3)(y - 5)$

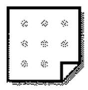
32. $(6 - x)(2x - 3)$

LESSON 8.3

Slope

You will need:

geoboards 

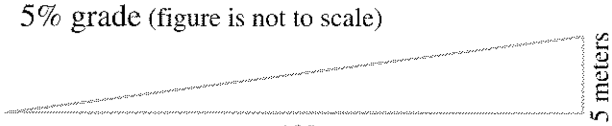
dot paper 

GRADE AND SLOPE

Steep roads sometimes have a sign indicating how steep they are. For example, the sign may say **5% Grade**. This means that you gain 5 units of altitude (the *rise*) for every 100 units you move in the horizontal direction (the *run*).

1. On a 5% grade, how many units of altitude do you gain for every
   a. 200 units you move in the horizontal direction?
   b. 25 units you move in the horizontal direction?
   c. 1 unit you move in the horizontal direction?

5% grade (figure is not to scale)



In math a 5% grade is called a *slope* of 0.05.

2. If the slope is 0.05, how many units do you move in the horizontal direction for every
   a. 30 units you gain in altitude?
   b. 0.05 units you gain in altitude?
   c. 0.5 units you gain in altitude?

3. The figure above is not to scale.
   a. What is the actual slope illustrated? (Use a ruler to measure the rise and the run.)
   b. Is it more or less steep than a 0.05 slope?

4. A sign in the mountains says **6% Grade. Trucks Use Low Gear.** Explain what a 6% grade is. Use the words *slope, rise,* and *run* in your answer.

5. In a nonmountainous area, the steepest grade allowed on a freeway is 4%. With this grade, how many meters of altitude do you gain per
   a. kilometer traveled in the horizontal direction? (A kilometer is 1000 meters.)
   b. meter traveled in the horizontal direction?

6. If you are climbing a mountain road with grade 5.5%, and you gain 1000 ft in altitude, how many miles have you traveled? (There are 5280 feet in a mile.)

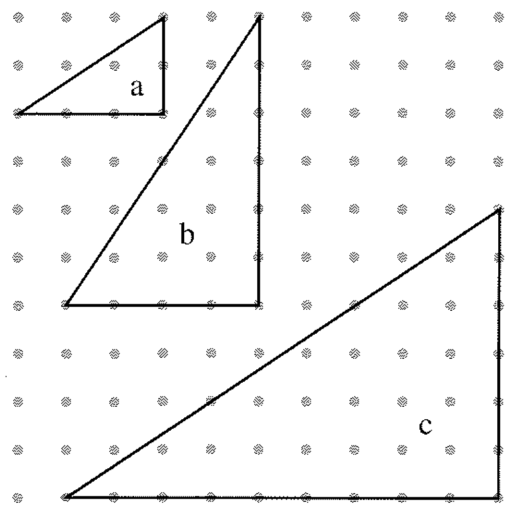
Definition: Slope is defined as the ratio of rise to run. $\text{slope} = \frac{\text{rise}}{\text{run}}$

7. a. How many units of altitude do you gain for every 100 units traveled on a horizontal road?
   b. What is the grade of a horizontal road?
   c. What is the slope of a horizontal line?

A horizontal road has grade 0. This is because no matter how much you move in the horizontal direction, you do not gain any altitude. The rise is 0 for any run. For example, for a run of 1, the slope is 0/1 which equals 0.

GEOBOARD SLOPE

The figure shows three geoboard right triangles. The side opposite the right angle in a right triangle is called the *hypotenuse.*



8. Find the slope of each hypotenuse in this figure.

9. How would you use the slope to find which hypotenuse is steeper? Which hypotenuses have the same steepness?

10. Two of the hypotenuses in the figure have the same slope. Explain why someone might make a mistake and believe all three have the same slope.

Do not use horizontal or vertical lines in problems 11-16. Start your lines at the origin.

11. a. What is the smallest slope you can find on a geoboard? Express it as a decimal.
b. Sketch a right triangle, like the ones above, to illustrate it.

12. Repeat problem 11 for the greatest possible geoboard slope.

13. Find a line having slope 1, and sketch several right triangles for it.

14. Find every possible geoboard slope that is a whole number.

15. Find every possible geoboard slope that is greater than 1 and less than 2. Express your answers as decimals.

16. Find every possible geoboard slope that is greater than 0.5 and less than 1. Express your answers as decimals.

SLOPES FROM COORDINATES

You may make a right triangle on your geoboard to help you answer the following questions.

17. What is the slope of the line joining
a. (0, 0) and (4, 5)?
b. (1, 1) and (5, 6)?
c. (0, 1) and (4, 5)?
d. (1, 0) and (5, 6)?

18. What is the slope of the line joining
a. (0, 0) and (8, 10)?
b. (0, 2) and (8, 10)?
c. (2, 3) and (3, 5)?
d. (4, 6) and (6, 10)?

For problem 19, you cannot use the geoboard.

19. What is the slope of the line joining
a. (23, 34) and (65, 54)?
b. (1.2, 3.4) and (5.6, 7.89)?

20. Generalization Explain how to find the slope of the line joining (a, b) and (c, d).

ROLLER COASTERS

Abe and Bea disagree about which roller coaster is steeper, the Plunge of Peril or the Drop of Death.

"The Plunge of Peril," according to the ad for the Great American Super-Park, "drops you 111 feet in seconds, with a mere 20 feet of horizontal displacement."

Abe and Bea have a photograph of themselves standing in front of the Drop of Death. They measured the roller coaster on the photograph, and got a drop of 10.1 cm for a run of 1.8 cm.

21. Use what you know about slope to help them decide which roller coaster is steeper. Explain your method.

22. Project The Plunge of Peril and the Drop of Death were invented for this lesson. Find the slopes of some real roller coasters.

DISCOVERY **SLUMBER THEORY**

Number theory is the branch of mathematics that studies whole numbers and their properties. It has been the source of many challenging problems over the centuries. Slumber theory is a silly offshoot of number theory.

The key concept of slumber theory is that any whole number can be *sliced* into a sequence of whole numbers.

Example: 365 can be sliced in four different ways:

3 | 6 | 5; 36 | 5; 3 | 65; or 365.

(Note that the slices are indicated by a vertical slash. Note also that in slumber theory, not slicing is considered a form of slicing.)

23. How many ways are there to slice a four-digit number?

A number is *slime* if it can be sliced into a sequence of primes.

Examples: 5 is slime, since it is already prime. 2027 is slime (2 | 02 | 7)
4,155,243,311 is slime
(41 | 5 | 5 | 2 | 43 | 3 | 11)

24. Which one of the following numbers is slime?

a. 12 b. 345 c. 6789

25. 2 is the only even prime. Find the first three even slimes.

26. There are no prime squares. Find the first two slime squares.

27. There are no prime cubes. Find the first two slime cubes.

28. 2 and 3 are the only consecutive numbers that are both prime. Find the first three pairs of consecutive numbers that are both slime.

29. There is no triple of consecutive numbers that are all prime. Find the first two triples of consecutive numbers that are all slime.

30. Find the smallest number that is slime in more than one way. (In other words, it can be sliced into two different sequences of primes.)

31. Find the smallest number that is slime in more than two ways.

A number is a super-slime if you get a sequence of primes no matter how you slice it.

Example: 53 is a super-slime since 53 and 5 | 3 are both sequences of primes.

32. Find all the super-slimes.

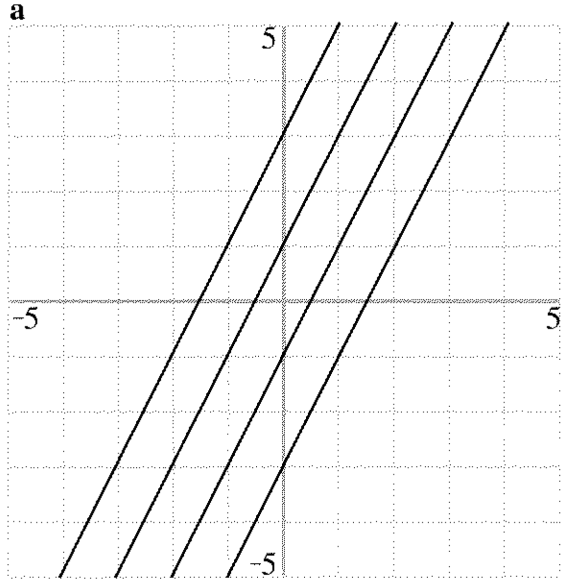
LESSON 8.4

Linear Functions

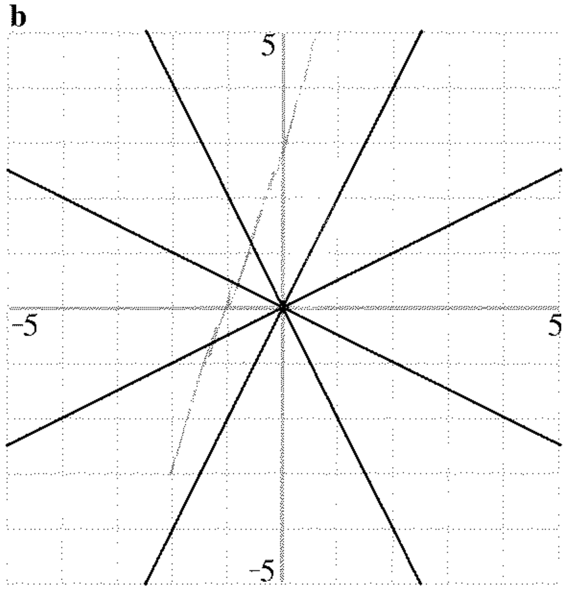
You will need:
graph paper

1. Exploration For problems (a-b), find the equations of lines that will create the given design.

a



b



THE SLOPE OF A LINE

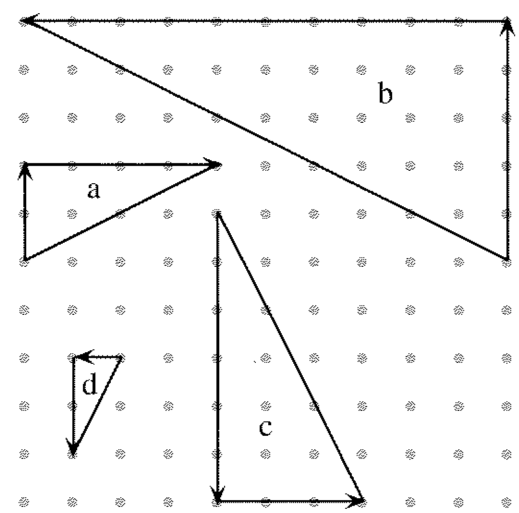
Definitions: The *rate of change* of a function is defined as a ratio between the change in y and the change in x.

$$\text{rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

In the Cartesian plane, a change in y-coordinates is called a *rise.* A change in x-coordinates is called a *run.*

The *slope of a line* is the ratio obtained when you divide the rise by the run. If you move from left to right, the run is positive. From right to left, it is negative. If you move up, the rise is positive. Moving down, it is negative.

The figure shows right triangles for slopes 0.5, -0.5, 2, and -2.



2. Match the slope to the triangle by finding the rise and the run as you move from one end of the hypotenuse to the other

a. in the direction of the arrows;

b. in the opposite direction.

c. Do you get the same answers both ways?

3. What can you say about the slope of a line if, when you follow the line *from left to right,*
   a. it goes up?
   b. it goes down?
   c. it goes neither up nor down?

4. Find two (x, y) pairs that satisfy each equation. Use them to graph the line. Label the two points, and use them to find the slope.
   a. $y = 1.5x + 3$
   b. $y = -1.5x + 3$
   c. $y = 2x + 3$
   d. $y = -3x + 3$

5. Think of the line with equation $y = 3x + 3$.
   a. Predict its slope.
   b. Check your prediction by graphing.
   c. For this function, when x increases by 1, by what does y increase?

6. Repeat problem 5 for $y = -2x + 3$.

7. How is the coefficient of x related to the slope?

THE y-INTERCEPT OF A LINE

8. For each of these equations, find the y-intercept.
   a. $y = 0.5x + 3$
   b. $y = 0.5x - 3$
   c. $y = 0.5x$
   d. $y = 0.5x + 1.5$

One way to find the y-intercept of a function is to graph it, and see where the graph meets the y-axis. Another way is to remember that *on the y-axis, the x-coordinate is 0.* In other words, all points on the y-axis are of the form (0, ___). So to find the y-intercept of a function, it is enough to substitute 0 for x, and find the value of y.

For each of these linear functions, answer the following questions. Graph the functions if you need to check your answers.
   a. When $x = 0$, what is y?
   b. When x increases by 1, by how much does y increase? (If y decreases, think of it as a negative increase.)
   c. What are the slope and y-intercept?

9. $y = x + 2$
10. $y = -4 - 3x$
11. $y = -x$
12. $y = 9$
13. $y = \frac{6x - 7}{8}$
14. $y = -2(x - 3)$

SLOPE AND y-INTERCEPT

Definition: $y = mx + b$ is called the *slope-intercept form* for the equation of a line.

For each equation below, tell whether it is in slope-intercept form.
   a. If it is, name m and b.
   b. If not, put it in slope-intercept form, then name m and b.

15. $y = 5x - 6$
16. $y = -4(x - 7)$
17. $y = \frac{5x - 6}{3}$
18. $y = \frac{x - 7}{-4}$
19. $y = 3(5x - 6)$
20. $y = -4x - 7$
21. $y + 4 = x$
22. $y + x = 4$

23. Without graphing each pair of lines, tell whether or not their graphs would intersect. Explain.
   a. $y = 2x + 8$ $\quad$ $y = 2x + 10$
   b. $y = -2x + 8$ $\quad$ $y = 2x + 10$
   c. $y = -2$ $\quad$ $y = 10$
   d. $y = x/4$ $\quad$ $y = 0.25x + 10$
   e. $y = 2(5x - 3)$ $\quad$ $y = 10x$

24. For (a-c), give the equation of a line that satisfies the following conditions.

a. It passes through the point (0, -2), and goes uphill from left to right.

b. It passes through the origin and (4, -6).

c. It does not contain any point in the third quadrant, and has slope -1.5.

Compare your answers with your classmates' answers.

25. Write three equations of the form $y = mx + b$. For each one, tell how much x changes when y changes by:

a. 1; b. 5; c. K.

26. Did your answers to problem 25 depend on the parameter m, the parameter b, or both?

27. What can you say about the signs of the slope and y-intercept of a line that does not contain any points in:

a. the first quadrant?

b. the second quadrant?

c. the third quadrant?

d. the fourth quadrant?

28. Report Explain how to use the slope-intercept form to predict the slope and y-intercept of a line. Make sure you give examples as you answer the following questions.

- What is the value of y when $x = 0$?
- When x increases by 1, by how much does y increase?
- How about when x increases by d?
- If two lines are parallel, what do their equations have in common?
- If two lines meet on the y-axis, what do their equations have in common?
- How is the slope-intercept form useful for graphing lines quickly?

PREVIEW ***WHAT'S THE FUNCTION?***

29. Think of the line that has slope -2 and passes through (1, 4).

a. By graphing, find any other point on the line.

b. Look at the graph to find the y-intercept.

c. What is the equation of the line?

30. Graph the line $y = 2x - 5$. Then graph each line, (a-c), and find its slope, y-intercept, and equation.

a. any line parallel to $y = 2x - 5$

b. the line parallel to it that passes through the origin

c. the line parallel to it that passes through the point (1, 4)

8.A Slope-Intercept Form

HORIZONTAL AND VERTICAL LINES

1. REVIEW What is the equation of:
   a. a horizontal line through (2, 3)?
   b. a vertical line through (2, 3)?
   c. the x-axis?
   d. the y-axis?

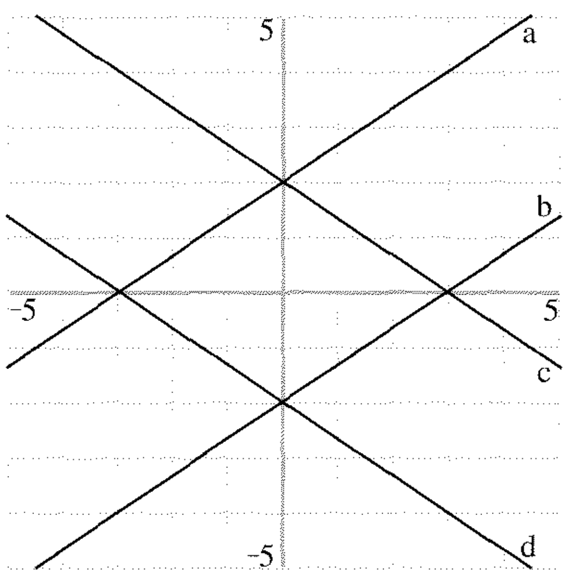
2. What is the slope of a horizontal line?

To find the slope of a vertical line, notice that the run is 0 for any rise. For example, for a run of 1, the slope should be 1/0, which is not defined. For this reason, vertical lines do not have a slope.

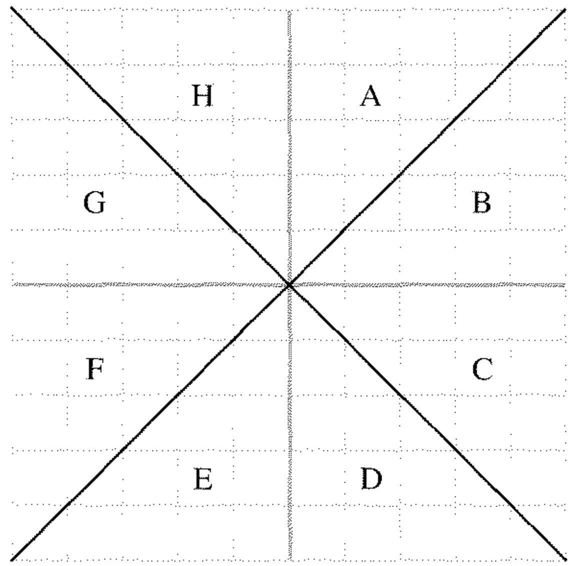
3. a. Explain why vertical lines do not have a y-intercept.
   b. Explain why the equations of vertical lines cannot be written in slope-intercept form.
   c. How does one write the equation of a vertical line?

FINDING m AND b

4. What are the equations of these lines, (a-d)?



5. a. What are the equations of the two lines in the graph below?
   b. What can you say about the equations of lines that pass through the origin and each of the regions A-H? (Your answers should be in the form: For lines in regions A and E, $b =$ ___ and m is between ___ and ___.)



6. What can you say about m if the graph is a very steep line, nearly vertical?

7. This table lists three points that all lie on one line. Find m and b without graphing.

x	y
-3	7
0	6
3	5

8. a. One of these tables lists three points which do not all lie on the same line. Which table is it? Explain how you can tell without graphing, by thinking about slope.

b. Find m and b for the other two tables.

x	y
-1	-7
1	1
3	9

x	y
-1	2
1	4
3	5

x	y
-1	8
1	0
3	-8

9. For equations (a-e), find m and b without graphing. (You may use graphing to check your answers.)

a. $y = -2$ b. $y = 9x$

c. $y = 2 - 3x$ d. $y = 4(5x - 6)$

e. $y = \frac{7x + 8}{9}$

10. **Report** Summarize what you know about slope-intercept form for linear functions. Illustrate your report with graphs and function diagrams. Use the words: equation, fixed point, focus, function, grade, graph, horizontal, linear, magnification, negative, parallel, parameters, positive, rate of change, ratio, slope, table, vertical, y-intercept.

LESSON 8.5

Ideal Population Growth

MATHEMATICAL MODELS

Exponents are useful for making mathematical descriptions of many kinds of growth, including population growth and spread of infectious disease. A mathematical description, or *mathematical model,* usually involves simplifying the real-world situation. Even though some of the idealized situations you study in this course may seem unrealistic, they will help you learn techniques that can be applied to more complicated real-world data.

Bacterial growth is one such situation. In research laboratories, bacteria used for biological studies are grown under controlled conditions. Although no real populations would grow as predictably as the ones described in this chapter, bacterial populations over short periods of time do approximate this kind of growth.

A DOUBLING POPULATION

1. A colony of bacteria is being grown in a laboratory. It contains a single bacterium at 12:00 noon (*time 0*), and the population is doubling every hour. How long do you think it would take for the population to exceed 1 million? 2 million? Write down your *guesses* and compare them with other students' guesses.

2. Make a table of values showing how the population in problem 1 changes as a function of time. Find the population one hour from now, two hours from now, etc. Extend your table until you can answer the questions asked in problem 1. How close were your guesses?

3. Add a third column to your table, writing the population each time as a power of 2.

4. What would the population be after x hours? (Write this as a power of 2.)

HOW MUCH MORE THAN? HOW MANY TIMES AS MUCH?

To determine the rate at which the population is increasing, we compare the populations at different times.

5. Compare the population after 8 hours with the population after 5 hours.
   a. *How much more* is the population after 8 hours? (Compare by subtracting.)
   b. *How many times* as much is it? (Compare by dividing.)
   c. Which of your answers in (a) and (b) can be written as a power of 2? What power of 2 is it?

6. Repeat problem 5, comparing the population after 7 hours with the population after 3 hours.

7. One of the questions, *How much more than?* or, *How many times as much?* can be answered easily with the help of powers of 2. Which question? Explain.

8. Make the comparisons below, answering the question: How many times as much? Write your answers as powers of 2.
   a. Compare the population after 12 hours with the population after 10 hours.
   b. Compare the population after 9 hours with the population after 4 hours.
   c. Compare the population after 4 hours with the population after 12 hours.

9. Compare each pair of numbers. The larger number is how many times as much as the smaller number? Write your answer as a power. In (d), assume x is positive.
   a. 2^6 and 2^9 b. 2^9 and 2^{14}
   c. 2^{14} and 2^6 d. 2^x and 2^{x+3}

A TRIPLING POPULATION

A colony of bacteria being grown in a laboratory contains a single bacterium at 12:00 noon (time 0). This population is *tripling* every hour.

10. Make the comparisons below, answering the question: How many times as much? Write your answers as powers of 3. (Hint: It may help to start by making a table showing how the population changes as a function of time.)

a. Compare the population after 12 hours with the population after 10 hours.

b. Compare the population after 9 hours with the population after 4 hours.

c. Compare the population after 4 hours with the population after 12 hours.

11. Compare each pair of numbers. How many times as much as the smaller number is the larger? Write your answer as a power. In (d), assume x is positive.

a. 3^6 and 3^9 b. 3^9 and 3^{14}

c. 3^{14} and 3^6 d. 3^x and 3^{x+5}

12. By what number would you have to multiply the first power to get the second power? Write your answer as a power.

a. $3^5 \cdot \_\_ = 3^{15}$ b. $3^8 \cdot \_\_ = 3^{15}$

c. $3^{11} \cdot \_\_ = 3^{15}$ d. $3^0 \cdot \_\_ = 3^{15}$

MULTIPLYING AND DIVIDING POWERS

In a power, the exponent tells how many times the base has been used as a factor. For example, 4^2 means $4 \cdot 4$, and 4^3 means $4 \cdot 4 \cdot 4$, therefore:

$$4^2 \cdot 4^3 = (4 \cdot 4) \cdot (4 \cdot 4 \cdot 4) = 4^5.$$

Use this idea to multiply powers.

13. Write the product as a power of 3.

a. $3^7 \cdot 3^3 =$ b. $3^5 \cdot 3^5 =$

c. $3^8 \cdot 3^2 =$ d. $3^8 \cdot 3^0 =$

14. Write the product as a power of 5.

a. $5^4 \cdot 5^3 =$ b. $5^4 \cdot 5^6 =$

c. $5^4 \cdot 5^9 =$ d. $5^0 \cdot 5^0 =$

When you divide, the quotient tells you how many times as much the numerator is than the denominator. For example, $4^5/4^2$ means *what times 4^2 equals 4^5?* Since $4^2 \cdot 4^3 = 4^5$, you have $4^5/4^2 = 4^3$. Use this idea to divide powers.

15. Write the quotient as a power of 2.

a. $\frac{2^{11}}{2^6}$ b. $\frac{2^6}{2^3}$

c. $\frac{2^{11}}{2^3}$ d. $\frac{2^{11}}{2^0}$

16. Write the quotient as a power of 3.

a. $\frac{3^7}{3^5}$ b. $\frac{3^6}{3^4}$

c. $\frac{3^{x+2}}{3^x}$ d. $\frac{3^{11}}{3^0}$

17. Use what you have learned in this lesson to find x.

a. $5^x \cdot 5^3 = 5^9$

b. $2^3 \cdot x^4 = 2^7$

c. $\frac{8^{66}}{8^x} = 8^{54}$

d. $\frac{10^{x+3}}{10^4} = 10^{a+1}$

18. Summary

a. Describe the patterns you found in multiplying and dividing powers.

b. Give examples to show how patterns can make it easier to multiply and divide powers.

c. In each multiplication and division problem, 15-17, *the bases of the powers are the same.* Does the pattern you described in (a) work if the bases are not the same? Explain, using examples.

19. Generalization Use the patterns you found in this lesson to rewrite each expression as a single power.

a. $5^x \cdot 5^y$ b. $a^x \cdot a^y$

c. $\frac{3^y}{3^x}$ d. $\frac{6^{x+5}}{6^x}$

e. $6^x \cdot 6^x$ f. $6^0 \cdot 6^x$

LESSON 8.6

Comparing Populations

EXPONENTIAL GROWTH

Three populations of bacteria are being grown in a laboratory. At time 0: Population A had 10 bacteria; Population B had 100 bacteria; and Population C had 300 bacteria. All three double every hour.

1. Complete the table below to show how the three populations increase as a function of time for the first six hours of growth after time 0.

Population

Time	A	B	C
0	10	100	300
1			

The populations are doubling, which means they are being repeatedly multiplied by 2. Powers of 2 provide a good shorthand for writing the populations.

2. Make another table of the populations of A, B, and C for the first six hours of growth after time 0. This time, use multiplication and a power of 2 to write each population. (Example: For A, the population after four hours is $10 \cdot 2^4$.)

3. Write the expressions for the populations of A, B, and C after:
   a. x hours;
   b. $x + 3$ hours.

Definitions: This kind of growth is called *exponential growth.* Exponential growth involves *repeated multiplication* by a number. To describe exponential growth, we specify the *starting population* and the *rate of growth.*

For example, if the starting population is 4 and the population triples every hour, this table shows how the population changes as a function of time.

Time	Population	Exponential Expression
0	4	$4 \cdot 3^0$
1	$4 \cdot 3 = 12$	$4 \cdot 3^1$
2	$4 \cdot 3 \cdot 3 = 36$	$4 \cdot 3^2$
3	$4 \cdot 3 \cdot 3 \cdot 3 = 108$	$4 \cdot 3^3$
x	$4 \cdot 3 \cdot ... = ?$	$4 \cdot 3^x$

Generalizations

4. Write an expression for the population after six hours of growth
   a. if the starting population is 100 and the population is tripling every hour;
   b. if the starting population is 100 and the population is being multiplied by r every hour;
   c. if the starting population is p and the population is being multiplied by r every hour.

5. Write an expression for the population after x hours of growth for each situation in problem 4.

SAME POPULATION, DIFFERENT TIME

6. The population of B after five hours is $100 \cdot 2^5$.
   a. Find the population of B at 8 hours, 11 hours, and 14 hours. By how much is the population being multiplied over each three-hour period?
   b. Compare the population of B after x hours with its population after $x + 3$ hours by simplifying this ratio.
   $$\frac{10 \cdot 2^{x+3}}{10 \cdot 2^x}$$

7. The population of A at six hours is $10 \cdot 2^6$.
   a. Compare the population of A after x hours with its population after $x + 5$ hours by simplifying this ratio.
   $$\frac{10 \cdot 2^{x+5}}{10 \cdot 2^x}$$
   b. Check your answer to part (a) by comparing the population of A at 6 hours, 11 hours, and 16 hours.

8. Simplify these ratios.
   a. $\frac{400 \cdot 2^7}{400 \cdot 2^3}$ b. $\frac{100 \cdot 2^{15}}{100 \cdot 2^8}$

9. Simplify these ratios. It may help to substitute values for x and look for a pattern.
   a. $\frac{400 \cdot 2^{x+7}}{400 \cdot 2^x}$ b. $\frac{100 \cdot 2^{3x}}{100 \cdot 2^x}$

10. Solve for x. $\frac{35 \cdot 2^{x+6}}{35 \cdot 2^x} = 2^x$

DIFFERENT POPULATIONS, SAME TIME

11. a. Use the tables you made in problems 1 and 2 to compare the size of A with the size of B at several times. In each case, B is how many times as large? Does this ratio increase, decrease, or remain the same as time goes on?
    b. Repeat part (a), comparing C with B.

12. Simplify these ratios.
    a. $\frac{400 \cdot 2^x}{200 \cdot 2^x}$ b. $\frac{10^0 \cdot 2^{x+4}}{500 \cdot 2^{x+4}}$

13. Solve for x. $\frac{300 \cdot 2^a}{x \cdot 2^a} = 30$

DIFFERENT POPULATIONS, DIFFERENT TIMES

14. Compare these populations using ratios.
    a. B at 10 hours and A at 3 hours
    b. C at 3 hours and A at 6 hours
    c. A at 12 hours and B at 7 hours
    d. C at 1/2 hour and A at 1 hour

15. Compare these populations using ratios.
    a. B at x hours and A at $x + 2$ hours
    b. C at h hours and A at $2h$ hours
    c. A at h hours and B at $h - 5$ hours

16. Simplify these ratios.
    a. $\frac{400 \cdot 2^{x+4}}{25 \cdot 2^x}$ b. $\frac{10 \cdot 2^{4x}}{150 \cdot 2^x}$

17. Solve for x.
    a. $\frac{30 \cdot 2^{a+4}}{x \cdot 2^a} = 60$
    b. $\frac{300 \cdot 2^{a+3}}{x \cdot 2^a} = 24$

POPULATION PROJECTIONS

In 1975 the population of the world was about 4.01 billion and was growing at a rate of about 2% per year. People used these facts to project what the population would be in the future.

18. Copy and complete the table, giving projections of the world's population from 1976 to 1980, assuming that the growth rate remained at 2% per year.

Year	Calculation	Projection (billions)
1976	4.01 + (0.02)4.01	4.09

19. Find the ratio of the projected population from year to year. Does the ratio increase, decrease, or stay the same?

20. There is a number that can be used to multiply one year's projection to calculate the next. What is that number?

21. Use repeated multiplication to project the world's population in 1990 from the 1975 number, assuming the same growth rate.

22. Compare your answer to problem 21 with the actual estimate of the population made in 1990, which was about 5.33 billion.

a. Did your projection over-estimate or under-estimate the 1990 population?

b. Was the population growth rate between 1975 and 1990 more or less than 2%? Explain.

23. At a growth rate of 2% a year, how long does it take for the world's population to double?

REVIEW *FACTORING COMPLETELY*

Example: $16 - 4x^2$ is a difference of two squares, so it can be factored:

$$(4 - 2x)(4 + 2x).$$

However, each of the binomials can be factored further, like this:

$$2(2 - x) \cdot 2(2 + x) = 4(2 - x)(2 + x)$$

Here is another way to factor the same expression:

$$4(4 - x^2) = 4(2 - x)(2 + x).$$

The final expression is the same one we got using the first method. It cannot be factored any further, so we say we have *factored completely*.

Factor each expression completely.

24. $3t^2 - 27s^2$

25. $5x^2 - 180$

26. $x^3y - xy^3$

LESSON 8.7

Percent Increase

You will need:

graph paper

AN ALGEBRA TUTOR'S SALARY

Bea did so well in algebra that she got a job as an algebra tutor. Her starting salary, as she had no experience, was $10 per week.

1. As Bea got more experience, her salary increased. She got a raise of $1 per week. Copy and complete the table for the first ten weeks that Bea worked.

Weeks	Salary	Amount increase	Percent increase
0	$10		
1	$11	$1	10
2	$12	$1	9
3	$13	$1	8.33

2. a. Explain how to calculate the number in the last column.

b. Explain why the number in the last column decreases each week.

3. Compare Bea's original salary with her salary for the tenth week.

a. What was the total amount of increase in her salary?

b. What percent of her original salary is this total increase? (This is the total *percent increase.*)

c. What percent of her original salary is her salary in the tenth week? (Your answer should be a number greater than 100. Why?)

Abe also got a job as an algebra tutor. He heard that Bea was getting a weekly raise of $1. Since $1 is 10% of $10, Abe asked for a weekly raise of 10%. The first week Bea and Abe both got the same raise.

4. Copy and complete the table for the first ten weeks that Abe worked.

Weeks	Salary	Amount increase	Percent increase
0	$10		
1	$11	$1	10
2	$12.10	$1.10	10
3	$13.31	$1.21	10

5. a. Explain how to calculate the numbers in the third column of the table above.

b. Explain why the numbers in the third column increase each week.

6. Repeat problem 3 for Abe's salary.

7. On the same pair of axes, make graphs of Abe's and Bea's weekly salaries as a function of weeks of experience.

8.

a. Each week's salary for Bea can be obtained from the previous week's salary by *adding* a number. Find this number and use it to write an equation that gives Bea's salary (S) as a function of weeks of experience (W).

b. Each week's salary for Abe can be obtained from the previous week's salary by *multiplying* by a number. Find this number, experimenting with your calculator if necessary, and use it to write an equation that gives Abe's salary as a function of weeks of experience.

9. 🔑

a. Write each equation you wrote on the graphs it belongs to.

b. Compare the graphs. Which is straight? Which is curved?

c. Which function describes linear growth? Which describes exponential growth?

10. Repeat the analysis you did for Abe's and Bea's salaries if Bea's raise were \$2 and Abe's raise were 20%.

EQUATIONS WITH PERCENTS

A state has 5% sales tax. If you paid \$12.60 for something, including tax, what was the price without tax? If the price without tax is x, and the increase due to tax is 0.05 of x, then

$$x + 0.05x = \$12.60.$$

11. 🔑 Remember that x can be written $1x$.

a. Combine like terms on the left side of the equation. (Or factor out the x.)

b. Then solve for x.

12. Solve for x.

a. $1.2x = 240$

b. $x + 0.4x = 18.2$

c. $x + 0.06x = 23.85$

d. $1.7x = 78.2$

13. Solve for x.

a. $(1.10)(1.10)x = 67.76$

b. $(1.10)(1.10)(1.10)x = 13.31$

The Skolar family eat out once a month. Usually they take turns figuring out the tip, also called the *gratuity*.

14. At one restaurant, they ordered food totaling \$35.95 and received a bill for the total amount they owed. The total was \$43.86, and the bill said "tax and gratuity included." Sue wrote this equation.

$$35.95 + p(35.95) = 43.86$$

a. Explain the equation. What does p represent?

b. Solve for p. Is your answer reasonable? Discuss.

15. Another night the Skolar family had \$23.00 to buy dinner. Assuming they'd need 25% of the cost of the dinner to cover the tax and tip, Michael wrote this equation.

$$d + 0.25d = 23.00$$

a. Explain the equation. What does d represent?

b. Solve for d.

16. Now assume the Skolars had \$23.00 for their meal and needed only 20% of the cost of the dinner to cover the tax and tip. How much can their actual food order be? Write and solve the equation.

EQUATIONS AND THE PRICE OF WIDGETS

17. A certain retail store sells widgets at the wholesale price, plus a 35% markup. If the wholesale price is W, what is the retail price of the widget? Express your answer as a function of W in two ways: as an addition and as a multiplication.

18. The wholesale cost of widgets went up by 8.5%. If the old wholesale price was W, express as a function of W,

a. the new wholesale price;

b. the new retail price;

c. the retail price including a 5% sales tax.

19. After the price increase in the wholesale cost a certain customer purchased a widget at the retail store for $15.71, including tax.

a. What was the wholesale price on that widget?

b. How much would the customer have saved by buying a widget before the wholesale price increase?

REVIEW *SOLVING EQUATIONS*

20. Solve for x.

a. $\frac{3^x}{3^2} = 3^5$

b. $\frac{10^{2x-5}}{10^2} = 10^5$

c. $\frac{p^{x-3}}{p^2} = p^6$

REVIEW *EQUATIONS AND INEQUALITIES*

Use the techniques you have learned to solve these equations and inequalities. You can use trial and error, the cover-up method, tables, graphs, or the Lab Gear. Show your work.

21. $5y > 2y + 57$

22. $3s + 7 = 4 + 3s$

23. $3(m + 4) + 3(m - 4) = 54$

24. $7 + y = 7y$

25. $\frac{10x + 4}{6} + 7 = -4$

26. $\frac{4x}{5} = 2 - x$

27. $\frac{3}{3x} = \frac{7}{4x - 2}$

28. $(2p + 3)^2 = (4p - 2)(p - 8)$

29. $(2p - 1)(3p + 2) = (6p - 1)(p + 1)$

30. $\frac{x}{x + 1} = 2$

31. $\frac{5}{x} + \frac{x}{5} = 2$

LESSON 8.8

Percent Decrease

A CASHIER'S QUANDARY

Sherman's Department Store ran the following ad in the newspaper.

G.D. and Cal were working during the three-hour sale. At the end of the sale, they compared receipts and discovered that they had sold some of the same items, but they had charged customers different prices for them. They made the following table.

Original price	Cal charged	G.D. charged
\$139.99	\$78.39	\$70.00
\$49.95	\$27.97	\$24.98
\$18.89	\$10.57	\$9.44
\$5.29	\$2.96	\$2.65
\$179.00	\$100.24	\$89.50

1. **Exploration** How was Cal calculating the sale price? How was G.D. calculating the sale price? Explain, showing sample calculations. Who do you think was right, and why?

LATE PAPER POLICIES

Mr. Peters, an algebra teacher, has a *10% off* late paper policy. This means that for each day that a paper is late, the student receives 90% of the credit that he or she would have received the day before. For example, if you turned in a perfect paper (assume a score out of 100) one day late, you would receive $(0.90)(100) = 90$ as your score. If you turned the paper in two days late, you would receive $(0.90)(90) = 81$ as your score.

2. Copy and extend Mr. Peters's table to show the score you would receive on a perfect paper that is up to ten days late.

Mr. Peters's Late Policy

Days late	Score
0	100
1	90
2	81

3. a. Explain how you figured out the scores in the table. Show some sample calculations.
   b. After how many days would your score for a late paper drop below 50?
   c. Would your score ever reach 0? Explain.

Mr. Riley, another algebra teacher, has a *10 points off* policy. This means that you lose ten points for each day that your paper is late.

4. Copy and extend Mr. Riley's table to show the score you would receive on a perfect paper that is up to ten days late.

Mr. Riley's Late Policy

Days late	Score
0	100
1	90
2	80

5. a After how many days would your score for a late paper drop below 50?
b. Would your score ever reach 0? Explain.

6. Graph the data in the two tables showing how the score decreases as a function of the number of days late. Use the same axes for both graphs so that you can compare them.

7. Write an equation that gives your score (S) on a perfect paper as a function of the number of days late (D)
a. in Mr. Peters's class;
b. in Mr. Riley's class.

8.
a. One of the equations you wrote in problem 7 should have an exponent. (If it doesn't, check your work.) Which equation has an exponent, the *percent off* policy, or the *points off* policy?
b. Write each equation you wrote in problem 7 on the corresponding graph. Does the equation containing an exponent correspond to the straight graph or to the curved graph?

9. Compare Mr. Riley's policy with Mr. Peters's policy. Which one do you prefer, and why? Give reasons why some students might prefer one policy and some students another.

DISCOUNTER INTRODUCES REDUCTIONS!

A store offers a 5% discount to students. If something costs \$15.00 after the discount is taken, how much does it cost without the discount? You can use percent decrease and algebra to solve this problem. If the price before the discount is x, and the decrease due to the discount is $0.05x$, then

$$x - 0.05x = \$15.00.$$

10. Remember that x can be written $1x$.
a. Combine like terms on the left side of the equation. (Or factor the x.)
b. Then solve for x.

11. Solve for x.
a. $0.2x = 240$
b. $x - 0.8x = 18.2$
c. $x - 0.06x = 23.50$
d. $x - 0.75x = 22.5$

12. Solve for x.
a. $(0.75)(0.75)x = 11.25$
b. $(0.65)^3x = 4.12$

Look back at the ad for Sherman's Store.

13. a. If the clearance sale price is \$13.50, what was the original price, before the 30% discount?

b. If the original price was \$20.95, what is the 30% discount price?

14. Report Let x be the original price of an item. Write two algebraic expressions for the early-bird price, one that will give the amount Cal would charge, and one for the amount G.D. would charge. Explain how you figured out these two expressions. Show that they work, by substituting the prices from the table into the expression.

REVIEW ***RATE OF CHANGE***

15. Find a function $y = mx + b$ for which

a. y increases when x increases;

b. y increases when x decreases;

c. y never increases.

16. Find a function $y = mx + b$, with m positive, for which y changes

a. faster than x;

b. more slowly than x;

c. at the same rate as x.

17. $y = x^7$ and $y = 2^x$ are having a race. When $x = 1$, $x^7 = 1$ and $2^x = 2$, so $y = 2^x$ is ahead. When $x = 3$, $x^7 = 2187$ and $2^x = 8$, so $y = x^7$ is ahead. As x gets larger and larger, who will win the race? Use your calculators and make a table to find out.

THINKING WRITING

8.B Simple and Compound Interest

Money in a savings account usually earns either *simple* or *compound* interest. For example, suppose you invest \$100 and earn 5% interest per year. If you earn *simple* interest, you will earn \$5 for every year that the money is invested, since 5 is 5% of 100. If you earn *compound* interest, you will earn \$5 for the first year the money is invested. In the next year, if you keep the entire \$105 in the bank, you will earn 5% interest on \$105. In other words, compound interest pays you interest on the interest as well as on the original investment.

The table shows what would happen to your investment in both cases for the first few years.

Total account balance, with:

Year	Simple interest	Compound interest
0	100	100
1	105	105
2	110	110.25
3	115	115.76

1. a. With simple interest, your account balance for each year can be obtained by *adding* a certain amount to the amount from the previous year. Find this amount.

b. With compound interest, your account balance for each year can be obtained by *multiplying* by a certain amount each year. Find this amount.

2. Write two equations (one for simple interest and one for compound interest) giving the account balance as the function of the year for:

a. 5% interest on the amount \$100;

b. 12% interest on the amount \$100;

c. 12% interest on the amount \$500.

3. Report Write a report comparing simple and compound interest. Your report should include, but not be limited to, the following:

- Equations for simple and compound interest that give the account balance as a function of time invested. Show how to change the equations if you change the amount of money invested or the interest rate. Explain how you figured out the equations.
- A comparison of how the amount in the account grows in each case. Which grows linearly and which grows exponentially? Explain how you know.
- An analysis of an example: Choose an amount to invest and an interest rate, and make a table or graph comparing the amount you would have in the account with simple and with compound interest. Assume you leave the money and the interest in the account for 25 years. Use a graph to illustrate.

4. Find a formula for the difference in the account balance after n years for two accounts that start with an original investment of s dollars at p percent interest, if one account earns simple interest and the other earns compound interest.

5. Say you have some money invested at 7% compound interest. How many *months* does it take for your investment to double? (Find a formula, then use decimal exponents on your calculator to find out what fraction of a year past a whole number of years it will take.)

LESSON 8.9

Equal Powers

In this lesson, use only whole number exponents.

1. **Exploration** The number 64 can be written as a power in at least three different ways, as 2^6, 8^2, or 4^3.
   a. Find some numbers that can be written as powers in two different ways.
   b. Find another number that can be written as a power in three different ways.

POWERS OF 3 AND 9

2. Using your calculator if necessary, try to find a power of 3 that is equal to each power of 9 below. If any are impossible, say so. Fill in the exponent.
   a. $9^2 = 81 = 3^?$
   b. $9^5 = 59049 = 3^?$
   c. $9^{10} = 3^?$
   d. $9^0 = 3^?$

3. Using your calculator if necessary, try to find a power of 9 that is equal to each power of 3 below. If any are impossible, say so. Fill in the exponent.
   a. $3^8 = 6561 = 9^?$
   b. $3^5 = 243 = 9^?$
   c. $3^{14} = 9^?$
   d. $3^0 = 9^?$

4.
   a. Can every power of 9 be written as a power of 3? If so, explain why. If not, show some that can and some that can't, and explain the difference.
   b. Can every power of 3 be written as a power of 9? If so, explain why. If not, show some that can and some that can't, and explain the difference.

POWERS OF 2, 4, 6, AND 8

5. Find two powers of 2 (other than 64) that can be written as powers of 8.

6. If the same number is written as both a power of 2 and a power of 8, how do the exponents compare? Explain and give examples.

7. Find at least three powers of 2 that can be written as powers of 4. Compare the exponents and describe what you notice.

8. Find at least two powers of 2 that can be written as powers of 16. Compare the exponents and describe what you notice.

9.
   a. Which powers of 2 can be written as powers of 8? Explain, giving examples.
   b. Which powers of 8 can be written as powers of 2? Explain, giving examples.
   c. Find the smallest number (besides 1) that can be written as a power of 2, a power of 4, and a power of 8. Write it in all three ways. How do you know that it is the smallest?

10. Can you find a number that can be written as a power of 2, a power of 4, and a power of 6? If so, find it. If not, explain why it is impossible.

WRITING POWERS USING DIFFERENT BASES

11. Write each number as a power using a smaller base.

a. 8^2	b. 27^3	c. 25^3
d. 16^4	e. 49^2	f. 2^0

12. Write each number as a power using a larger base.

a. 3^2 b. 9^4 c. 4^8
d. 5^8 e. 6^6 f. 95^0

13. If possible, write each number as a power using a different base. (Do not use the exponent 1.) If it is not possible, explain why not.

a. 3^4 b. 3^3
c. 4^5 d. 3^5

14. Repeat problem 13 for these numbers.

a. 5^4 b. 5^3
c. 25^2 d. 26^4

15. Summary If you exclude the exponent 1, when it is possible to write a number in two or more ways as a power? Does it depend on the base, the exponent, or both? Explain. (Give examples of some equivalent powers and of numbers that can be written as powers in only one way.)

16. Generalization Fill in the exponents.

a. $9^x = 3^?$ b. $4^x = 2^?$
c. $8^x = 2^?$ d. $16^x =$
e. $25^x =$

A POWER OF A POWER

Since $9 = 3^2$, the power 9^3 can be written as $(3^2)^3$. The expression $(3^2)^3$ is a *power of a power* of 3.

17. a. Write 25^3 as a power of a power of 5.
b. Write 8^5 as a power of a power of 2.
c. Write 9^4 as a power of a power of 3.

There is often a simpler way to write a power of a power. For example:

$$(3^5)^2 = (3^5)(3^5)$$
$$= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)$$
$$= 3^{10}$$

18. a. Show how $(2^5)^3$ can be written with one exponent as a power of 2.
b. Write $(3^4)^2$ as a power of 3.

19. Is $(4^5)^3$ equal to 4^8, to 4^{15}, or to neither? Explain.

Generalizations

20. Fill in the exponents.

a. $(x^2)^3 = x^?$ b. $y^4 = (y^2)^?$
c. $y^{10} = (y^5)^?$ d. $y^6 = (y^3)^?$
e. $(x^4)^3 = x^?$

21. Fill in the exponents.

a. $(y^2)^x = y^?$ b. $(y^3)^x = y^?$
c. $(x^4)^y = x^?$ d. $y^{ax} = (y^x)^?$

The generalization you made in problem 21 is one of the *laws of exponents.* It is sometimes called the *power of a power law:*

$$(x^a)^b = x^{ab}, \text{ as long as } x \text{ is not } 0.$$

22. Explain how the ideas you discussed in problem 15 are related to the power of a power law.

LESSON 8.10

Working With Monomials

The product of the monomials $3x^2$ and $9x^4$ is also a monomial. This can be shown by using the definition of exponentiation as repeated multiplication.

$$3x^2 = 3 \cdot x \cdot x \text{ and } 9x^4 = 9 \cdot x \cdot x \cdot x \cdot x$$

so

$$3x^2 \cdot 9x^4 = 3 \cdot x \cdot x \cdot 9 \cdot x \cdot x \cdot x \cdot x = 27x^6$$

1. Find another pair of monomials whose product is $27x^6$.

2. **Exploration** If possible, find at least two answers to each of these problems. Write $27x^6$ as:

a. the product of three monomials

b. the sum of three monomials

c. a monomial raised to a power

d. the quotient of two monomials

e. the difference of two monomials

PRODUCT OF POWERS

The monomial $48x^9$ can be written as a product in many different ways. For example, $16x^6 \cdot 3x^3$ and $12x^5 \cdot 4x^3 \cdot x$ are both equal to $48x^9$.

3. Write $48x^9$ in three more ways as a product of two or more monomials.

4. Write $35x^4$ as a product in which one of the factors is

a. a third-degree monomial;

b. a monomial with a coefficient of 7;

c. $5x^0$;

d. $35x^3$.

5. Write $7.2 \cdot 10^8$ in three ways as the product of two numbers in scientific notation.

6. Write x^5 in three ways as a product of two or more monomials.

7. **Generalization** Study your answers to problem 6. Then fill in the exponent.

$$x^a \cdot x^b = x^?$$

Explain.

8. If possible, write each expression more simply. If it is not possible, explain why not.

a. $3x^5 \cdot 6x^4$ b. $x^5 \cdot y^7$

c. $y^7 \cdot y^3$ d. $4a^4 \cdot 9a^3$

The generalization you made is one of the laws of exponents. It is sometimes called the *product of powers* law. It says that

$$x^a \cdot x^b = x^{a+b}, \text{ as long as } x \text{ is not } 0.$$

However, notice that it works only when the bases are the same.

POWER OF A PRODUCT

The expression $x^4 \cdot y^4 \cdot z^4$ is the product of three powers. Since the bases are not the same, we cannot use the product of powers law. However, notice that since the exponents are the same, it is possible to write a product of powers as a single power

$$\begin{aligned} x^4y^4z^4 &= x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot z \cdot z \cdot z \cdot z \\ &= xyz \cdot xyz \cdot xyz \cdot xyz \\ &= (xyz)^4 \end{aligned}$$

9. Write $16a^2b^2$ as the square of a monomial. (Hint: First rewrite 16 as a power.)

10. Write p^3q^3 as the cube of a monomial.

11. If possible, write each expression as a single power. If it is not possible, explain why not.

a. $32n^5m^5$ b. x^2y^3

c. $(2n)^7 \cdot (3m)^7$ d. $(ab)^4 \cdot (bc)^4$

The generalization you used above is another of the laws of exponents. It is sometimes called the *power of a product* law. It says that

$x^a y^a = (xy)^a$, as long as x and y are not 0.

However, notice that it works only when the exponents are the same.

12. Write without parentheses.

a. $(6y)^2$ b. $(3xy)^4$
c. $(5xyz)^3$ d. $(2x)^3$
e. $(2xy)^3$ f. $(2xyz)^3$

13. Write $64x^3y^6z^9$ as the cube of a monomial.

POWER OF A RATIO

14. Write 49/25 as the square of a ratio.

Study this example.

$$\left(\frac{x}{y}\right)^3 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x \cdot x \cdot x}{y \cdot y \cdot y} = \frac{x^3}{y^3}$$

This law of exponents is called the *power of a ratio* law. It says that

$\frac{x^a}{y^a} = \left(\frac{x}{y}\right)^a$, as long as x and y are not 0.

However, notice that it works only when the exponents are the same.

15. Write as a power of a ratio.

a. $8x^3/y^6$ b. $16x^4/x^{10}$

16. Write as a ratio of monomials.

a. $(5x/7z)^9$ b. $(2xy/yz)^2$

RATIOS OF MONOMIALS

Consider the ratio $6x^5/4x^7$. By multiplying numerator and denominator by x, you can get the equivalent ratio $6x^6/4x^7$. Or you can get an equivalent ratio in lowest terms by noticing that

$$\frac{6x^5}{4x^7} = \frac{3}{2x^2} \cdot \frac{2x^5}{2x^5} = \frac{3}{2x^2}.$$

17. Explain the example above.

18. Write in lowest terms.

a. $8x^8/6x^9$ b. $7x^7/5x^4$

In some cases, a ratio can be simplified to a monomial. For example,

$$\frac{150x^6}{50x^4} = 3x^2.$$

19. a. Explain this example.

b. Write $3x^2$ as a ratio of monomials in three other ways.

20. Write $12y^3$ as a quotient of two monomials in which

a. one is a fourth-degree monomial;
b. one has a coefficient of 5;
c. one is a monomial of degree 0.

21. Write $1.2 \cdot 10^4$ in three ways as the quotient of two numbers in scientific notation.

22. a. Write x^5 as a ratio in three ways.

b. Find three ratios equivalent to $1/x^5$.

23. **Generalization** Study your answers to problem 22. Compare the situations in (a) and (b). Explain how to simplify a ratio whose numerator and denominator are powers of x.

24. Fill in the exponent, assuming $p > q$.

$$\frac{x^p}{x^q} = x^?$$

25. Write these ratios in lowest terms.

a. $3x^5/6x^4$ b. x^5/y^7
c. y^3/y^7 d. $45a^4/9a^3$

26. Write as a power of 6. $\frac{6^{x-5}}{6^{5-x}}$

SOLVING EQUATIONS

Solve for x.

27. a. $\frac{5^{2x}}{5^x} = 5^7$ b. $\frac{(7^2)^x}{7^4} = 7^6$

28. a. $\frac{(3 \cdot 5)^3}{108 \cdot 5^x} = \frac{1}{20}$ b. $\frac{3^3 \cdot 4^7}{3 \cdot 4^x} = \left(\frac{3}{4}\right)^2$

29.

a. $\frac{3 \cdot 4^{6p}}{9 \cdot 4^x} = \frac{1}{3 \cdot 4^{4p}}$ b. $\frac{15h^x}{12h^a} = \frac{5}{4h^6}$

LESSON 8.11

Negative Bases, Negative Exponents

RECIPROCALS

In previous lessons, we have considered only whole number exponents. Does a negative exponent have any meaning? To answer this, consider these patterns.

$3^4 = 81$	$(1/3)^4 = 1/81$
$3^3 = 27$	$(1/3)^3 = 1/27$
$3^2 = 9$	$(1/3)^2 = 1/9$
$3^1 = 3$	$(1/3)^1 = 1/3$
$3^0 = 1$	$(1/3)^0 = 1$
$3^{-1} = ?$	$(1/3)^{-1} = ?$

1. a. Look at the powers of 3. How is each number related to the number above it? Following this pattern, what should the value of 3^{-1} be?

b. Now look for a pattern in the powers of 1/3. As the exponent increases, does the value of the power increase or decrease? Following this pattern, what should the value of $(1/3)^{-1}$ be?

c. Compare the values of 3^{-1}, 3^1, $(1/3)^1$ and $(1/3)^{-1}$. How are they related?

d. Use the pattern you found to extend the table down to 3^{-4} and $(1/3)^{-4}$.

Another way to figure out the meaning of negative exponents is to use the product of powers law. For example, to figure out the meaning of 3^{-1}, note that:

$$3^{-1} \cdot 3^2 = 3^1$$
$$3^{-1} \cdot 9 = 3$$

But the only number that can be multiplied by 9 to get 3 is 1/3, so 3^{-1} must equal 1/3.

2. Find the value of 3^{-1} by applying the product of powers law to $3^1 \cdot 3^{-1}$.

3. Use the same logic to find the value of:
a. 3^{-2}; b. 3^{-x}.

4. Are the answers you found in problem 3 consistent with the pattern you found in problem 1? Explain.

5. **Summary** People who have not studied algebra (and, unfortunately, many who have) think that 5^{-2} equals a negative number, such as -25.

a. Write a convincing argument using the product of powers law to explain why this is not true.

b. Show how to find the value of 5^{-2} using a pattern like the one in problem 1.

6. a. Show that $5x^2$ and $5x^{-2}$ are not reciprocals, by showing that their product is not 1.

b. Find the reciprocal of $5x^2$.

MORE ON EXPONENTIAL GROWTH

A bacterial culture doubles every hour. At this moment it weighs 10 grams.

7. What did it weigh

a. 1 hour ago?

b. 2 hours ago?

c. x hours ago?

8.

a. Explain why the weight of the bacteria culture x hours from now is given by

$$W = 10 \cdot 2^x.$$

b. Explain the meaning of substituting a negative value for x.

9. Show your calculations, using the equation in problem 8, to find out:
a. how much it will weigh in three hours;
b. how much it weighed three hours ago.

In 1975 the world population was about 4.01 billion and growing at the rate of 2% per year.

10. If it continued to grow at that rate, write a formula for the world population after x years.

If it had been growing at the same rate before 1975, we could estimate the population in previous years by using negative values of x in the formula.

11. Use your calculator to find the value of $(1.02)^4$ and its reciprocal, $(1.02)^{-4}$.

12. Show your calculations using the equation in problem 10 to estimate the population in:
a. 1971; b. 1979.

13. Assume the world population had been growing at this rate since 1925.
a. Estimate the world population in 1925.
b. Compare this number with the actual world population in 1925, which was about 2 billion. Was the population growth rate between 1925 and 1975 more or less than 2%? Explain.

RATIO OF POWERS

Negative exponents often arise when simplifying ratios of monomials.

This law of exponents is sometimes called the *ratio of powers* law:

$$\frac{x^a}{x^b} = x^{a-b}\text{, as long as } x \text{ is not } 0.$$

However, notice that it works only when the bases are the same.

Examples:

$\frac{x^6}{x^7} = x^{6-7} = x^{-1}$ or $\frac{1}{x^1}$

$\frac{x^{3a}}{x^{5a}} = x^{3a-5a} = x^{-2a}$ or $\frac{1}{x^{2a}}$

14. Simplify.
a. $4x^6/5x^7$ b. $2x^8y^3/2xy$
c. y^3/y^7 d. $45a/9a^5$

15. Simplify these ratios.
a. $\frac{400a^5}{25a^2}$ b. $\frac{400x^3}{200x^8}$
c. $\frac{3m^6}{9m^3}$ d. $\frac{9R^a}{3R^a}$

16.
a. Write as a power of 4, $4^{3+x}/4^{3-x}$.
b. Write as a power of 7, $7^{5x-5}/7^{5x-6}$.

17. Solve for x.
a. $\frac{7^4}{7^{x+2}} = 7^3$
b. $\frac{3 \cdot 5^{x+2}}{12 \cdot 5^2} = \frac{1}{20}$

18. Divide without using your calculator. Then, if your answer is not already in scientific notation, convert it to scientific notation.
a. $\frac{4.2 \cdot 10^5}{3.0 \cdot 10^2}$ b. $\frac{3.0 \cdot 10^4}{1.5 \cdot 10^6}$
c. $\frac{1.5 \cdot 10^3}{3.0 \cdot 10^6}$ d. $\frac{9 \cdot 10^a}{3 \cdot 10^b}$

OPPOSITES

The expression $(-5)^3$ has a negative base. This expression means *raise -5 to the third power.* The expression -5^3 has a positive base. This expression means *raise 5 to the third power and take the opposite of the result.*

19. Which of these expressions represent negative numbers? Show the calculations or explain the reasoning leading to your conclusions.

-5^3 $(-5)^3$ -5^2 $(-7)^{15}$ $(-7)^{14}$

-5^{-3} $(-5)^{-3}$ -5^{-2} $(-7)^{-15}$ $(-7)^{-14}$

20.

a. Is $(-5)^n$ always, sometimes, or never the opposite of 5^n? Explain, using examples.

b. Is -5^n always, sometimes, or never the opposite of 5^n? Explain, using examples.

EARLY PAPERS

Ms. Kem has a policy that penalizes students for turning in papers late. Her students are trying to convince her to give them extra points for turning in their papers early. Some students propose a policy based on adding points. Others propose one based on increasing by a percentage.

21. If you were her student, what kind of early paper policy would you propose?

22. Using your policy, what would your score be, if your paper were x days early?

REVIEW *WHICH IS GREATER?*

Or are they equal?

23. a. $x - 0.30x$ b. $0.70x$

24. a. $(0.70)(0.70)x$ b. $x - 0.50x$

25. a. $(0.90)(0.90)(0.90)x$
b. $x - 0.10x - 0.10x - 0.10x$

REVIEW *EQUATION SOLVING*

Solve for x.

26. a. $(0.85)(0.85)(0.85)(0.85)x = 18.79$
b. $x - 0.2x = 160$
c. $0.80x = 500$

27. $\frac{50b^3}{xb} = 2b^2$

28. $\frac{20a^{m+1}}{10a^m} = 2a^x$

REVIEW *WHAT'S THE FUNCTION?*

29. Find the slope of the line that goes through each pair of points. Then find the equation for the line. (Hint: A sketch may help.)

a. (0, 1) and (2, 3)
b. (0, 4) and (0.5, -6)
c. (0, 7) and (-0.8, 0.9)

30. In problem 29
a. how did you find the y-intercept?
b. how did you find the slope?

31. Find the equation for the line
a. having slope 0.9, passing through (2, -1);
b. having slope 3.4, passing through (6.7, 9);
c. passing through (8, 2) and (1.3, -5.4).

LESSON 8.12 Small and Large Numbers

1. Using a power of ten, write the reciprocal of each number.

   a. 10^2 b. 10^4 c. 0.001

SMALL NUMBERS IN SCIENTIFIC NOTATION

Any decimal number can be written in many ways as a product of a decimal number and a power of 10. For example, 43,000 can be written:

$$0.43 \cdot 10^5$$
$$4.3 \cdot 10^4$$
$$43 \cdot 10^3$$
$$430 \cdot 10^2$$
$$4300 \cdot 10^1$$

2. Write 43,000 as a product of a decimal number and

   a. 10^0; b. 10^{-1}; c. 10^{-2}.

3. a. Write 0.065 in three ways as a product of a decimal number and a power of 10. At least one way should use a negative exponent.

   b. Write 0.065 in scientific notation. (Remember that scientific notation requires multiplying a number greater than or equal to 1 and less than 10 by a power of 10.)

4. Which of these numbers would require a negative exponent when written in scientific notation? Explain why.

   0.0123 0.123 12.3 1230

5. How can you tell by looking at a decimal number whether or not it will require a negative exponent when it is written in scientific notation?

RECIPROCALS

Al and Abe, having nothing else to do, were arguing about reciprocals. Abe said, "If 10^{-4} is the reciprocal of 10^4, then $2.5 \cdot 10^{-4}$ is the reciprocal of $2.5 \cdot 10^4$." Al said, "I can prove that you're wrong by finding their product."

6. If $2.5 \cdot 10^{-4}$ is the reciprocal of $2.5 \cdot 10^4$, what should their product be?

7. Settle the argument between Al and Abe. If Abe has not found the correct reciprocal of $2.5 \cdot 10^4$, find it for him. Explain.

8. Find an approximation for the reciprocal of $4.6 \cdot 10^{-6}$. Give your answer in scientific notation.

UNITS AND RECIPROCALS

9. Dick walks at the rate of about five miles in one hour. What fraction of an hour does it take him to walk one mile?

10. Stanley can run about ten miles in one hour. What fraction of an hour does it take him to run one mile?

11. A snail travels at the rate of 0.005 miles per hour. How many hours does it take the snail to slither one mile?

Notice that your answers to problems 9-11 are the reciprocals of the rates given. This is not a coincidence. In each case, the rate is given in *miles/hour* and you are asked to find *hours/mile.* Since the units are reciprocals, the rates will also be reciprocals.

12. Sound travels through air at the rate of $1.088 \cdot 10^3$ feet per second at sea level. How long does it take sound to travel one foot?

13. Sound travels much faster through granite than through air. Its speed is about $1.2906 \cdot 10^4$ feet per second. How long does it take sound to travel one foot through granite?

UNITS IN THE METRIC SYSTEM

The metric system of measurement is based on powers of ten. Prefixes indicating powers of ten are used for all measurements within the metric system. Conversion between units is easy, since it involves multiplying by powers of ten.

Example: The prefix *kilo* means to multiply the basic unit of measure by 10^3, or 1000. A kilogram is 1000 grams, a kilometer is 1000 meters, and so on. This table lists some of these prefixes.

To Multiply by	Prefix
10^{12}	tera-
10^9	giga-
10^6	mega-
10^3	kilo-
10^2	hecto-
10^1	deka-
10^0	—
10^{-1}	deci-
10^{-2}	centi-
10^{-3}	milli-
10^{-6}	micro-
10^{-9}	nano
10^{-12}	pico-

14. Express the size of each object in terms of a more appropriate unit of measurement.
    a. A redwood tree is 80,023 millimeters high.
    b. A protozoan is 0.0000002 kilometers in diameter.
    c. A football player weighs 95,130 grams.

15. At the San Andreas fault in Northern California, the ground is moving about $5 \cdot 10^{-5}$ kilometers per year. How long will it take to move one kilometer?

16. If hair grows at the rate of about 10^{-8} miles per hour, how long would it take your hair to reach ankle length? (Why is this problem harder than the previous ones?)

THINKING WRITING 8.C Applying the Laws of Exponents

Tina overslept and had to skip breakfast, so she didn't do very well on her math test. Besides, she had forgotten to study the laws of exponents. In fact, she missed *all* the problems.

Test Name: *Tina A.*

Exponents

Instructions: Simplify. Your answer should have only one exponent. Not all are possible.

a. $2^4 \cdot 3^4 = 5^4$

b. $3^{15} + 6^{15} = 9^{15}$

c. $3x^2 \cdot 2x^3 = 5x^5$

d. $\frac{x^7}{y^3} = \left(\frac{x}{y}\right)^4$

e. $10x^5 \cdot 8x^9 = 80x^{45}$

f. $(2x)^7 = 2x^7$

g. $12x^3 \cdot 4y^7 = 48(xy)^{10}$

h. $\left(\frac{3^7}{3^5}\right)^3 = 1^6$

i. $(3x^2)^3 = 3x^5$

j. $x^3 + x^2 = x^5$

1. Summarize the five laws of exponents given in Lessons 9, 10, and 11.

2. Correct Tina's test. For each problem, write the correct answer. If one or more of the laws of exponents was used, tell which law (or laws) was used. If the expression cannot be simplified, say so.

3. Take Tina's make-up test for her. Be careful! (Remember, make-up tests are always harder.)

Make-up Test Name: *Tina A.*

Laws of Exponents

Instructions: Show all work leading to your answer.

a. Write without parentheses: $(4x^2y^3z)^3$

Perform each operation, and *if possible* write the result as a power of 5.

b. $5^{11} - 5^9$

c. $5^{x+3}/5^x$

d. $5^5 \cdot \_\_ = 5^{15}$

e. $5^7 + 5^3$

If possible, write as a power of 12.

f. $3 \cdot 4^3$

g. $(3 \cdot 4)^5$

h. $2 \cdot 6^8$

i. $2^8 \cdot 6^8$

j. $2^8 \cdot 6^5$

Which expression is not equal to the other two?

k. 3^{100} 6^{75} 9^{50}

l. $(y^2)^4$ $(y^4)^2$ y^4y^2

m. a^7 $a^3 + a^4$ $a^3 \cdot a^4$

Write the opposite of the reciprocal of $(1/2)^5$

n. using a negative exponent;

o. using a positive exponent.

Essential Ideas

POPULATION OF NORTH AMERICA

The table shows the estimated population of North America from 1650 to 1950.

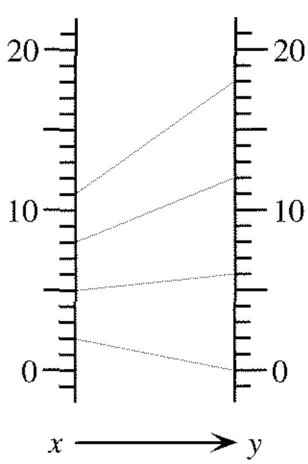
Year	Population (thousands)
1650	5000
1750	5000
1850	39,000
1900	106,000
1950	219,000

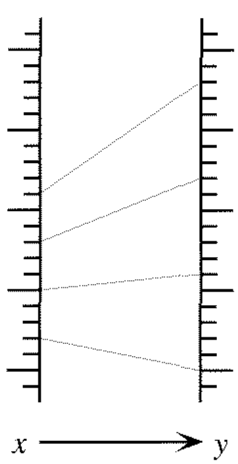
1. What is the population increase in each 100-year period?

2. Graph the data.
   a. What is the meaning of slope for this data?
   b. Is the slope constant or does it increase or decrease? Explain.

3. Estimate the population of North America in the year
   a. 1800; b. 2000.
   Explain how you arrive at your estimates.

SAME DIAGRAM, DIFFERENT SCALE





4. Make an in-out table for the function diagram on the left. What is the function illustrated?

5. The function diagram on the right is the same, except that the number lines are not labeled. Copy the diagram, and put labels on it, using the same scale on both the x- and y-number lines. Make an in-out table, and find the function .

6. Repeat problem 5 two times. For each diagram, make an in-out table and find the function.

7. **Summary**
   a. For the functions you found in problems 4-6, when x increases by 1, what does y increase by? Does it depend on the scale you used?
   b. Compare the functions you found in problems 4-6. How are they the same? How are they different? Explain.

SLOPE AND INTERCEPT

The following questions are about the graph of $y = mx + b$.

8. Describe the line if $b = 0$ and
   a. $m > 1$ b. $0 < m < 1$
   c. $m = 0$ d. $-1 < m < 0$
   e. $m < -1$

9. In which quadrants does the line lie if
   a. $b > 0, m > 0$? b. $b < 0, m > 0$?
   c. $b > 0, m < 0$? d. $b < 0, m < 0$?

10. How would lines be the same or different if
   a. they have the same value for b and different values for m?
   b. they have the same value for m and different values for b?

LINEAR AND EXPONENTIAL GROWTH

11. Two populations are growing exponentially. At time 0, both have populations of 100. If one is growing twice as fast as the other, how do their populations compare after:

a. 2 hours? b. 3 hours?

c. x hours?

12. Report A recent college graduate was offered a job with a salary of \$20,000 per year and a guarantee of a 5% raise every year. She was about to accept the job when she received another offer for an identical job with a salary of \$22,000 per year and a guarantee of a \$1200 raise each year. Explain how you would help her decide which job to accept.

LAWS OF EXPONENTS

13. If possible, write as a power of 4.

a. $2 \cdot 2^6$ b. $(2 \cdot 2)^6$

c. $2 \cdot 2^5$ d. $2^7 \cdot 2^5$

e. $2^5 \cdot 2^5$

14. If possible, write as a power of 6.

a. $2 \cdot 3^5$ b. $(2 \cdot 3)^5$

c. 36^7 d. 36^0

15. If possible, write as a power of 3.

a. $9 \cdot 3^5 \cdot 3^2 \cdot 3^0$

b. $9 \cdot 3^5 \cdot 3^2 \cdot 2^0$

c. $9 \cdot 3^5 \cdot 2^2 \cdot 2^0$

d. $81 \cdot (3^5)^4 \cdot 6^0$

16. If possible, write as a single monomial.

a. $8a^{12} - 2(3a^3)^4$

b. $\left(\frac{6t^3}{4}\right)^2 - t^5$

17. Find values of a, b, and c so that

a. $(a \cdot b)^c > a \cdot b^c$;

b. $(a \cdot b)^c = a \cdot b^c$;

c. $(a \cdot b)^c < a \cdot b^c$.

18. Find the number or expression that makes each equation true. Write your answer as a power.

a. $(3x)^4 =$ ____ $\cdot x^4$

b. $(5t)^3 =$ ____ $\cdot t^3$

c. $(12xy)^3 =$ ____ $\cdot (3xy)^3$

19. Simplify each ratio.

a. $(2x^5)/x^5$ b. $(2x)^5/x^5$

c. Explain why your answers to (a) and (b) are different.

20. Find the number that makes each equation true. Write your answer as a power.

a. $100 \cdot (2R)^5 =$ ______ $\cdot 100 \cdot R^5$

b. $20 \cdot (2x)^7 =$ ________ $\cdot 20 \cdot x^7$

c. $(2xyz)^{10} =$ ________ $\cdot (xyz)^{10}$

21. Find the number that makes each equation true. Write your answer as a power.

a. $100 \cdot (3R)^5 =$ ______ $\cdot 100 \cdot R^5$

b. $20 \cdot (3x)^7 =$ ________ $\cdot 20 \cdot x^7$

c. $(3xyz)^{10} =$ ________ $\cdot (xyz)^{10}$

22. Find the reciprocal. Check by showing that the product is 1.

a. $14x^3y^3$ b. $-3a^5$

c. $\frac{1}{3b^2}$

Because of variables in the exponents, these problems are more challenging.

23. Simplify.

a. $\frac{9 \cdot 10^{a+5}}{3 \cdot 10^a}$ b. $\frac{3 \cdot 10^{b+2}}{9 \cdot 10^b}$

c. $\frac{9 \cdot R^{a+5}}{3 \cdot R^a}$ d. $\frac{12 \cdot y^{b+2}}{10 \cdot y^b}$

24. Write as a power of 5.

a. $\frac{5^{2x-2}}{5^{x-5}}$ b. $\frac{5^{x-5}}{5^{2x-2}}$

25. Write as a power of 4. $\left(\frac{4^{3+x}}{4^{3-x}}\right)^3$

VERY SMALL NUMBERS

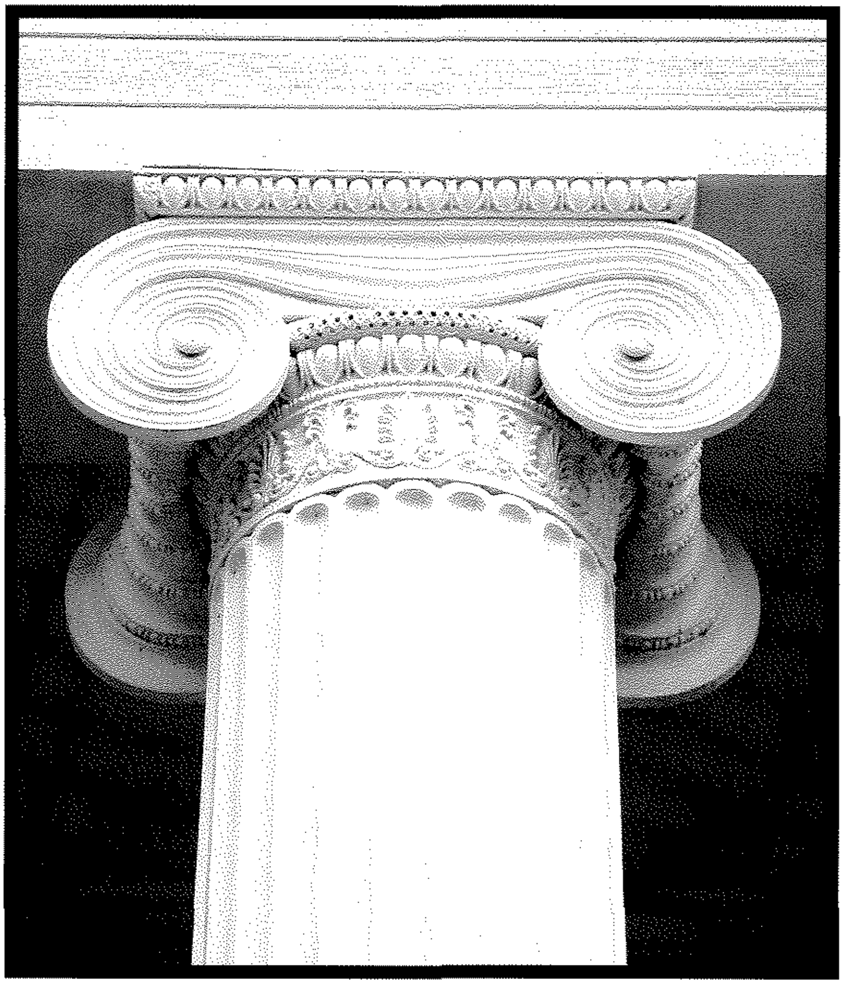
A proton weighs $1.674 \cdot 10^{-24}$ grams, an electron weighs $9.110 \cdot 10^{-28}$ grams, and Ann weighs 48 kilograms.

26. Which is heavier, a proton or an electron? How many times as heavy?

27. Ann weighs the same as how many
   a. electrons?
   b. protons?

28. The mean distance between the Earth and the sun is $1.50 \cdot 10^{11}$ meters. This length is called one *astronomical unit* (AU) and is a convenient unit for measuring distances in the solar system. The distance 10^{-10} meters is called one *angstrom* (after the Swedish physicist Anders Angstrom). It is a convenient unit for measuring atoms. How many angstroms are in one AU?

CHAPTER 9



The scroll-like spirals in the capital of a Greek column

Coming in this chapter:

Exploration There are four geoboard segments that start at the origin and have length 5. Find their endpoints. Use this to help you solve the following problem: If you know that two sides of a geoboard triangle are of length 5, what are the possible lengths for the third side?