Isomorphism

Definition: Two mathematical structures are isomorphic if their elements can be matched one-to-one in such a way that the results of operations are preserved.

Example: We can match the elements of the YZ group with the elements of the additive mod 6 group as follows:

$Y \to 2$
$Z \to 3$
$E \to 0$

Does this make sense so far? Remember that YYY = E, which corresponds to $2 \oplus 2 \oplus 2 = 0$. Likewise, ZZ = E, which corresponds to $3 \oplus 3 = 0$. For the isomorphism to work, we need to find the match for the other elements, and make sure the results of the operations are preserved. For example, YY should correspond to $2 \oplus 2$, so YY $\to$ 4.

1. Find the matches for YZ and YYZ. Do we have an isomorphism?

2. There is another isomorphism between those same groups, which starts with $Y \to 4$. Find the other matches.

3. Is the triangle symmetry group isomorphic to the YZ group? To the yz group?

4. Find an isomorphism between the “def” group of Glosian money and the additive calendar group.

5. Show that the multiplicative calendar group is isomorphic to the additive mod 6 group.

Fields

Definition: An algebraic system $\{S, +, \cdot\}$ consisting of a set S together with two operations + and ·, is called a *field* if it has the following properties.

$\forall$ a, b, c in S:

A1. Addition is associative: $a + (b + c) = (a + b) + c$

A2. Addition is commutative: $a + b = b + a$

A3. Zero: $\exists$ an element 0 in S such that $a + 0 = a$

A4. Opposite: $\exists$ an element -a such that $a + -a = 0$

M1. Multiplication is associative: $a(bc) = (ab)c$

M2. Multiplication is commutative: $ab = ba$

M3. One: $\exists$ an element 1 in S such that $1a = a$

M4. Reciprocal: if $a \neq 0$, $\exists$ an element $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$

D. Multiplication is distributive over addition: $a(b + c) = ab + ac$

1. Explain why the integers with + and · are not a field.
2. Explain why the rational numbers with + and · are a field.
3. Show that the set of numbers mod 5 with $\oplus$ and $\otimes$ is a field.
4. Show that the set of numbers mod 6 with $\oplus$ and $\otimes$ is not a field.
5. Is Calendar Math a field?

Adding Number Pairs

1. The whole numbers {0, 1, 2, …} with addition do not form a group. Why?

Think of the set of all ordered pairs of whole numbers, such as (2, 5). The operation + on ordered pairs is defined as follows: (a,b) + (c,d) = (a+c, b+d)

2. Calculate:
   a. (3,3) + (5,6)   b. (4,3) + (5,6)   c. (2,5) + (4,2)

Say that the two pairs (a,b) and (c,d) are *equivalent* when a+d = b+c.

3. a. Name some pairs that are equivalent to (2,5).
   b. Display them on a graph.

The set of all pairs that are equivalent to (2,5) is called an *equivalence class*.

4. Repeat #3 for (2,2) and (4,2), making sure that each equivalence class is easy to distinguish from the others on the graph.

5. What is the smallest pair that is equivalent to (5,9)? To (9,5)?

6. Choose two equivalence classes, A and B. For example, you might choose the class of (2,5) for A and the class of (4,2) for B. Add a number pair from A to a number pair from B. Try it again with other pairs from A and B. Are the results equivalent? Explain algebraically why they *must* be equivalent.

Let T be the set of equivalence classes as defined as above. Define ⊕ as follows: if A and B are equivalence classes in T, A⊕B is the equivalence class of the sum of an element from A and an element from B. For example, using the A and B from #6: (2,5)+(4,2)=(6,7), so A⊕B is the equivalence class of (6,7): {(0,1), (1,2), (2,3), …} ⊕ is *well defined* because as we showed in #6, the result of the operation does not depend on which representatives of the equivalence classes we choose.

7. Show that {T, ⊕} is a group.

8. Match the elements in T with the integers, and show that {T, ⊕} has the same structure as the integers with addition.

Multiplication on the number pairs is defined as follows: (a,b) · (c,d) = (ad+bc, ac+bd). ⊗ in T is defined in a similar way to ⊕ above from multiplication of number pairs.

9. Show that the definition of ⊗ makes sense.

10. Perhaps surprisingly, {T, ⊗} has the same structure as the integers with multiplication. Check that on some examples. Why does this work?

This approach is a way to create integers entirely from the whole numbers.

Inventing Rationals

1. The integers with addition and multiplication are not a field. Explain why.

Let P be the set of all ordered pairs of integers, such as (2, -5) or (0, 7), with the operations ⊕ and ⊗, defined as follows:

$$(a,b) \oplus (c,d) = (ad+bc, bd)$$
$$(a,b) \otimes (c,d) = (ac, bd)$$

2. Calculate:
   a. (4,3) ⊕ (-5,6)
   b. (-2,1) ⊗ (7,1)

3. Write as a single fraction:
   a. $\frac{a}{b} + \frac{c}{d}$
   b. $\frac{a}{b} \cdot \frac{c}{d}$

4. The system {P, ⊕, ⊗} is very much like a familiar system. Which one? Explain. (Hint: find a way to match elements between the two systems, and show that corresponding elements, when multiplied or added, yield corresponding elements.)

5. To match the familiar system, some elements must be removed from P. Which ones?

This approach allows us to define the rational numbers in terms of pairs of integers, like we defined the integers in terms of pairs of natural numbers.

6. Add an equivalence criterion to the rules above, so that two pairs that correspond to the same rational number are considered equivalent. The challenge is to write this strictly in terms of integers and their operations.

7. If you graph equivalence classes as defined in the previous problem, what do they look like?