Transformational Proof in High School Geometry

A guide for teachers and curriculum developers

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March 2018

The latest version of this document will always be at
http://www.MathEducationPage.org/transformations/

Preface: Rationale for Transformational Proof

Chapter 1: Isometries and Congruence

Chapter 2: Symmetry Definitions and Properties: Triangles and Quadrilaterals

Chapter 3: Proving Triangles and Quadrilaterals Satisfy Symmetry Definitions

Chapter 4: Dilation and Similarity

Chapter 5: Circles

Chapter 6: The Pythagorean Theorem

Appendix: List of Theorems in Chapters 1, 4, and 5

Lew Douglas passed away in 2019.

Preface: Transformational Proof Rationale

The Common Core State Standards for Mathematics (CCSSM) include a fundamental change in the geometry curriculum in grades 8 through 10: geometric transformations, not congruence and similarity postulates, are to constitute the logical foundation of geometry at this level. We propose an approach to triangle congruence and similarity, and more generally to geometric proof where advantageous, that is compatible with this new vision.

A pedagogical argument for this change: Congruence postulates are rather technical and far from self-evident to a beginner. In fact, many teachers introduce the basic idea of congruence by saying something like “If you can superpose two figures, they are congruent.” That is not very far from saying “If you can move one figure to land exactly on top of the other, they are congruent.” In other words, basing congruence on transformations is more intuitive than going in the other direction.

There are also mathematical arguments for the change. A transformational approach

1. offers deeper links between algebra and geometry because of its emphasis on functions (and thus composition of functions, inverse functions, fixed points, and so on.)
2. highlights the natural way transformations connect with complex numbers and matrices, so can immeasurably enhance the teaching of these topics in grades 11-12.
3. will give symmetry a greater role in school mathematics, which not only enhances geometric thinking and connects easily with art and nature, but also enhances student motivation.
4. makes it possible to discuss the similarity of curves (such as circles and parabolas), which could not be done under the traditional definition of similarity because it relies on equal angles and proportional sides.
5. paves the way for transformations of graphs and transformations using coordinates in intermediate algebra and beyond.
6. blends more naturally with dynamic geometry software, such as the geometry component of Desmos and GeoGebra. This is because the transformations tools are often useful in constructing dynamic special polygons.

One of the consequences of this change is the need for some clarity on how this new vision affects the logical structure of high school geometry. Our development so far attempts to fill that need by starting with a new set of definitions and assumptions, then using these to prove the usual (and some less usual) theorems. When we complete this development, meant solely for teachers and curriculum developers, we hope to begin work on materials for students. If you use this document as a guide for work with students, or if you have comments or questions, please get in touch: henri@MathEducation.page.

Beyond pedagogical and mathematical considerations, transformations are an integral part of everyday life. Flips and turns are built-in tools when drawing shapes on a computer. Enlargement or reduction on a computer screen or a copier is a dilation with the scale factor shown as a percent. When we see a photograph, we know that the people depicted in it are not really 2 inches high – their images have been dilated using a positive scale factor it are not really 2 inches high – their images have been dilated using a positive scale factor less than 1. Images on a movie screen have been dilated using a scale factor greater than 1. Flat maps are transformations of the surface of a sphere to a plane. As we look around us, more and more examples of transformations become apparent.

Note on Geometric Construction

The essential mathematical concept underlying geometric construction is *not* the use of straightedge and compass. Interesting versions of construction have been developed for straightedge and the collapsing compass, and for the compass alone, not to mention for pedagogical tools such as patty paper, Plexiglas mirrors, and of course interactive geometry software.

The essential concept underlying geometric construction is that of *intersecting loci*. The *locus* of a point is the set of all possible locations of that point, given the point's properties. The locus can be a line, a circle, or some other curve. If one knows two loci for a certain point, the point must lie at their intersection. In other words, given a figure, an additional point can be added to it in a mathematically rigorous way by knowing the locus (location) of the point in two different ways. Geometric construction is the challenge of finding such points and, in some cases, using them to define additional parts of the figure.

In the sequence we propose, arguments based on what we call the *construction postulates* (see Chapter 1) are needed for many proofs. One pedagogical consequence of this is that construction challenges should play an important role in teaching geometry. We favor an introduction using compass, straightedge, and patty paper, soon followed by work with interactive geometry software.

Note on Common Core compliance

The Common Core State Standards for Mathematics only require using transformations to justify the triangle congruence and similarity criteria. Beyond that, they do not specify whether proofs can or should use transformational approaches. The most traditional interpretation of the Standards would revert to traditional proofs based on congruent and similar triangles. The approach we take in this document is to use a transformational approach whenever possible. Classroom teachers and curriculum developers can situate themselves anywhere in this range. The best strategy is probably one that combines traditional and transformational approaches to proof in different cases, and also compares them in some instances. Note that we did not include theorems for which we could not find a suitable transformational alternative to the traditional proof.

More on Transformational Geometry

See http://www.mathedpage.org/transformations for related articles, and some curricular materials.

Chapter 1: Isometries and Congruence

Basic Transformational Definitions

Transformation of the Plane: A one-to-one function whose domain and range are the entire plane. (It is understood that we are working in the 2-D plane.)
Isometry (or rigid motion): A transformation of the plane that preserves distance and angles.
Symmetry of a Figure: An isometry for which the figure, taken as a whole, is invariant. Individual points in the figure need not be fixed.
Congruence: Two figures are *congruent* if one can be superposed on the other by a sequence of isometries. (Or: two figures are congruent if one is the image of the other under a composition of isometries.)
Note: Many traditional definitions are unchanged. For example, the perpendicular bisector of a segment is the line perpendicular to the segment through its midpoint.

New Definition of Parallel

It will simplify the statement of some theorems if we call lines that coincide parallel. This is especially useful for theorems involving translation. Therefore, we will say that two lines are parallel if they do not intersect or coincide. We will call traditional parallel lines "distinct parallel lines." Segments or rays are parallel if the lines that contain them are parallel. Parallel rays can point in the same or opposite directions.

Notation

We use the customary symbols for parallel, perpendicular, angle, and triangle, but we generally do not use symbols for segment, ray, or line. For example, we say *segment AB* or *line CD*. EF by itself can be any of these if the context makes it clear. As is customary, EF can also be the distance between E and F, which is also the length of segment EF. We sometimes use an arrow over one or more letters for a vector, but context usually makes it clear.
In general, we refer to the image of point P under a transformation as P', and likewise for lines and other figures.

Definitions of Basic Isometries

Reflection: A reflection in a line b maps any point on b to itself, and any other point P to a point P' so that b is the perpendicular bisector of segment PP'.
Rotation: Given a point O and a directed angle θ, the image of a point $P \neq O$ under a rotation with center O and angle θ is a point P' on the circle centered at O with radius OP, such that $\angle POP' = \theta$. The image of O is O. (θ is positive for a counterclockwise rotation, negative for clockwise.) A 180° rotation is also known as a *half-turn* or *reflection in a point* (the center of rotation).

Vector An arrow specifying distance and direction for which position doesn't matter. Two representatives of the same vector will be parallel, have the same length, and point in the same direction. $\vec{v} = \vec{w}$ means they are two representatives of the same vector.
Parallel Vectors: Vectors whose representatives lie on parallel lines. They do not need to have the same length.
Opposite Vectors: Two equal-length parallel vectors pointing in opposite directions. $-\vec{v}$ denotes the vector opposite $\vec{v}$.
Translation: Given a vector $\vec{v}$, the image of a point P under a translation by $\vec{v}$, is a point P' such that $\overrightarrow{PP'} = \vec{v}$.

Postulates

The following five assumptions are sufficient for the mathematically experienced, though a truly rigorous development would include the protractor postulate and others from Hilbert's set. When working with students or developing curriculum, many of the basic theorems proved below can be added to the set of assumptions because many students will think they are obvious.

1. The parallel postulate: Through a point outside a given line, one and only one line can be drawn parallel to the given line.
   Note: The parallel postulate has many equivalent forms. This one is generally credited to Playfair. See https://en.wikipedia.org/wiki/Parallel_postulate.
2. Reflection preserves distance and angle measure.

And the *construction postulates:*

3. Two distinct lines meet in at most one point.
4. A circle and a line meet in at most two points.
5. Two distinct circles meet in at most two points.

Pedagogical Note

In order to limit ourselves to a minimum number of postulates, we only assume that reflection is an isometry. To ensure the logical progression of this presentation, we prove that rotations and translations are isometries further down (see “Two Reflections” below). However, in the classroom it may be preferable to present all three as isometries early on, and allow students to find transformational proofs of Theorems 9-12 that rely on one or more of the three basic isometries. This can be done, say, using interactive geometry software.

Some theorems in this chapter are probably obvious to your students. Supplying formal proofs for them will likely be counterproductive, as students find it incomprehensible that such results require proof. This is true whether the proof is simple or complicated. We have found it more effective to reserve formal proof for less obvious theorems, such as those in subsequent chapters.

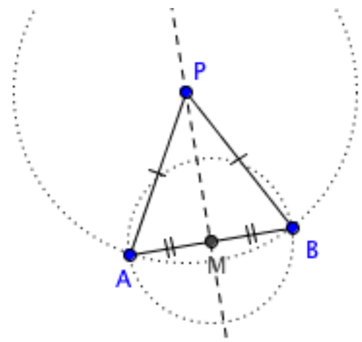
In this document, we use an asterisk (*) to mark theorems for which we discourage a formal discussion. For completeness and logical consistency, we offer proofs for these theorems below, but in the classroom, you can probably either assume those results or discuss them informally. Admittedly, this is a judgment call in each case.

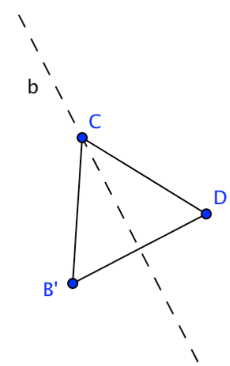
Requiring a minimal list of axioms and proving seemingly obvious results does not constitute an accessible introduction to proof. In fact, it is a rather sophisticated stance. Should you have an exceptional student who demands a rigorous treatment with a minimum number of assumptions, refer them to this document.

Basic Theorems

1. * If A' = B under a reflection, then B' = A.
   Proof: If l is the reflection line, then l is the perpendicular bisector of segment AB, which is the same as segment BA.
2. * Reflection preserves collinearity and betweenness.
   Proof: A, B, and C are collinear with B between A and C if and only if $AB + BC = AC$. Since reflection preserves distance, $A'B' + B'C' = A'C'$.
3. * If $A \rightarrow A'$ and $B \rightarrow B'$ under a reflection, segment AB must map onto segment $A'B'$.
   Proof: Because reflection preserves collinearity and betweenness, segment AB must map onto part or all of segment $A'B'$. Reflect any point P on segment $A'B'$ under the same reflection. $P' \rightarrow P$ by Theorem 1. So, any point on segment $A'B'$ is an image point and segment AB reflects onto the entire segment $A'B'$.
4. * Reflections map rays onto to rays and lines onto lines.
   Proof: The argument of Theorem 3 applies.
5. * Congruent segments have equal length. Congruent angles have equal measure.
   Proof: Isometries preserve segment length and angle measure.
6. * The corresponding sides and angles of congruent polygons have equal measure.
   Proof: Isometries preserve segment length and angle measure. (This is true for reflection by Postulate 2. It will be proved for rotation and translation later in this document.)
7. * There is a reflection that maps any given point P onto any given point Q.
   Proof: If $P = Q$, reflection in any line through P will do the job. If not, Q is the reflection of P across the perpendicular bisector of segment PQ.

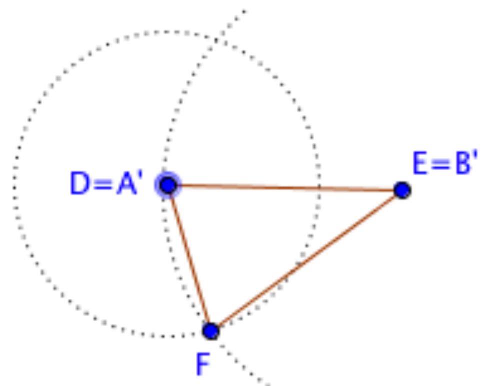
Triangle Congruence

8. A point P is equidistant from two points A and B if and only if it lies on their perpendicular bisector.
   

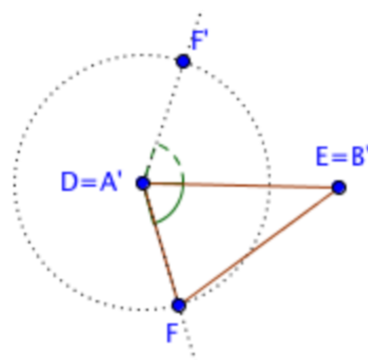
   Proof: If P is on the perpendicular bisector b of AB, then by the definition of reflection, A and B are images of each other and P is its own image in a reflection across b. So, $PA = PB$ since reflections preserve distance.
   Conversely, if $PA = PB$, we must show that P is on the perpendicular bisector of AB. Let M be the midpoint of AB. Reflect A across PM. Call the image A'.
   Since reflections preserve distance, A' must be on the circle centered at P, with radius PA, and on the circle centered at M, with radius MA. Because $PA = PB$ and M is the midpoint of AB, B must be on both circles as well.
   $A' \neq A$ because A is not on the reflection line, so $A' = B$, the other intersection point of the circles. Since PM is the perpendicular bisector of AA', it is the perpendicular bisector of AB. We conclude that P is indeed on the perpendicular bisector of AB. (An indirect proof is also possible if one assumes the triangle inequality.)
9. If two segments AB and CD have equal length, then one is the image of the other, with C the image of A and D the image of B, under either one or two reflections.
   

   Proof: Given $AB = CD$, by Theorem 7, we can reflect segment AB so that C is the image of A. Let B' be the image of B. If $B'=D$, that single reflection will do. If not, since reflections preserve distance, we have $CB'= AB = CD$. By Theorem 8, C is on the perpendicular bisector b of $B'D$. Therefore, a second reflection of CB' in b yields CD.
10. Equal length segments are congruent. If we combine this with Theorem 5, we have: Segments are congruent if and only if they have equal length.
    Proof: This is an immediate corollary of Theorem 9.

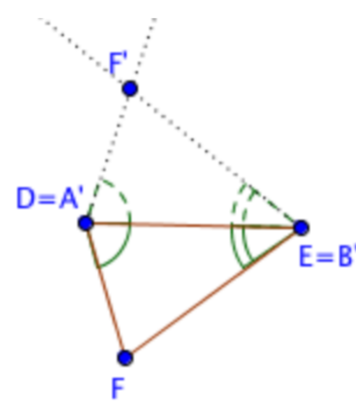
11. Congruence Criteria for Triangles
   a. SSS Congruence: If all sides of one triangle are congruent, respectively, to all sides of another, then the triangles are congruent.
   Proof: We are given ΔABC and ΔDEF with $AB = DE$, $BC = EF$, and $AC = DF$. By Theorem 9, we can superpose AB onto DE in one or two reflections. Because reflections preserve distance, C' (the image of C) must be at the intersection of two circles: one centered at D, with radius DF, the other centered at E, with radius EF. F, of course, is on both circles. If $C' = F$, we're done. If not, C' must be at the other intersection. But by Theorem 8, DE must be the perpendicular bisector of FC', so a reflection across DE superposes ΔABC onto ΔDEF.



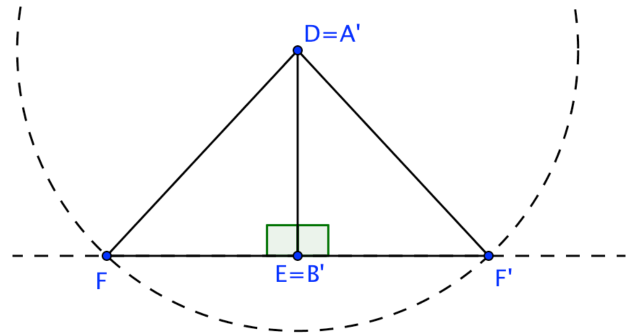
   b. SAS Congruence: If two sides of one triangle are congruent to two sides of another, and if the included angles have equal measure, then the triangles are congruent.
   Proof: We are given ΔABC and ΔDEF, with $AB = DE$, $BC = EF$, and $\angle B = \angle E$. By Theorem 9, we can superpose AB onto DE in one or two reflections. If $C' = F$, we're done. If not, reflect F across DE with image F'. Because reflections preserve distance and angle measure, C' must lie on the ray DF' and on the circle centered at D with radius DF. Therefore $C' = F'$, so a reflection across DE superposes ΔABC onto ΔDEF.



   c. ASA Congruence: If two angles of one triangle are congruent to two angles of another, and if the sides common to these angles in each triangle are congruent, then the triangles are congruent.
   Proof: We are given ΔABC and ΔDEF with $AB = DE$, $\angle A = \angle D$, and $\angle B = \angle E$. By Theorem 9, we can superpose AB onto DE in one or two reflections. If $C' = F$, we're done. If not, reflect F across DE with image F'. Since reflections preserves angle measure, C' must be on ray DF' and on ray EF'. It follows that $C' = F'$, so a reflection across DE superposes ΔABC onto ΔDEF.



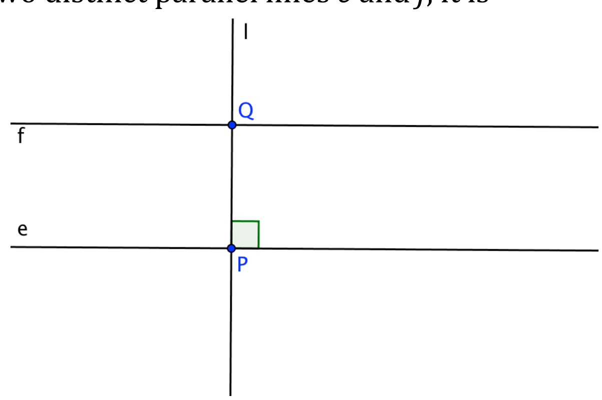
12. HL Congruence Criterion for Right Triangles: If the hypotenuse and one leg of one right triangle are congruent to the hypotenuse and one leg of another, then the right triangles are congruent.
    Proof: We are given ΔABC and ΔDEF with hypotenuse AC = hypotenuse DF and leg AB = leg DE. By Theorem 9, we can superpose AB onto DE in one or two reflections. Because $\angle ABC$ is a right angle and reflection preserves angle measure, $B'C' \perp A'B'$. But $\angle DEF$ is a right angle also and reflection preserves segment length, so C' lies on the line through E perpendicular to DE and on the circle with center D and radius DF. If $C' = F$, we're done. If not, reflect F across DE with image F'. C' must lie on ray EF' and on the circle with center D and radius DF. It follows that $C' = F'$, so a reflection across DE superposes ΔABC onto ΔDEF.



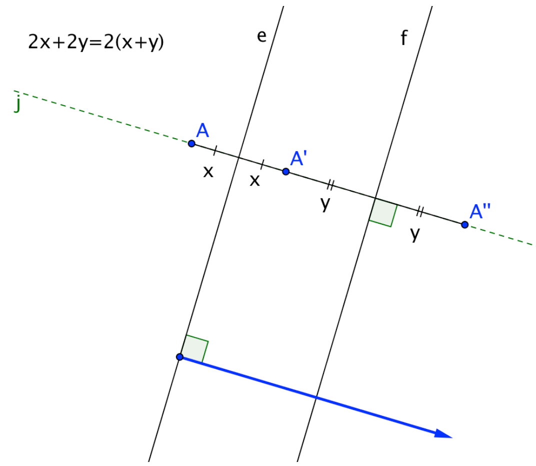
13. If two triangles are congruent, one can be superposed on the other by a sequence of at most three reflections.
    Proof: The proofs in Theorem 11 and 12 show this.
14. * Angles with equal measure are congruent. If we combine this with Theorem 5, we have : Angles are congruent if and only if they have equal measure
    Proof: Given $\angle A = \angle D$, construct ΔABC and ΔDEF with $AB = AC = DE = DF$. $\Delta ABC \cong \Delta DEF$ by SAS, so a sequence of reflections superimposes ΔABC onto ΔDEF. This implies that $\angle A$ maps onto $\angle D$.

Two Reflections

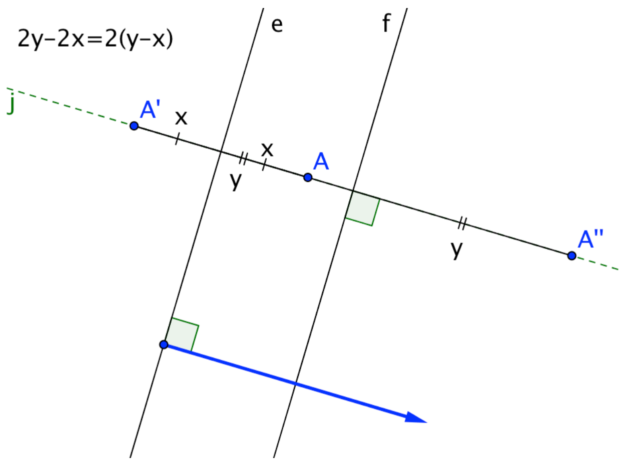
15. * If a line l is perpendicular to one of two distinct parallel lines e and f, it is perpendicular to the other.
    Proof: In the diagram, l is perpendicular to e at P and f intersects l at Q. Assume that f is not perpendicular to l. Reflect e and f in l. Because Q is on l, $Q' = Q$, and because e is perpendicular to l, $e' = e$. If f were not perpendicular to l, $f' \neq f$. f' can't also be parallel to e, because that would contradict the parallel postulate. So, e and f' intersect at a point R. Reflecting again in l would show that e and f intersect at R'. But this is impossible, because e and f are parallel. Therefore, our assumption is incorrect, and f must be perpendicular to l.



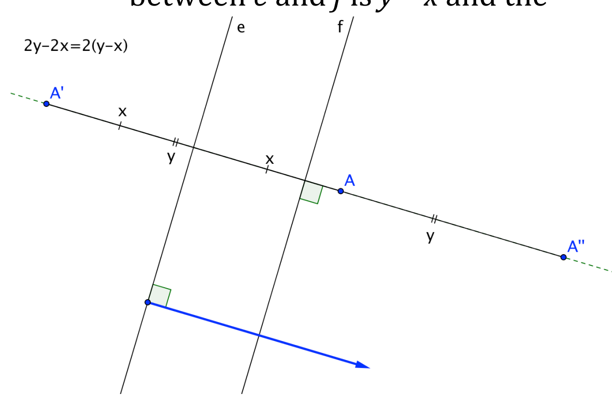
16. The composition of two reflections in parallel lines is a translation. The translation vector is perpendicular to the lines, points from the first line to the second, and has length twice the distance between the lines. This implies that any translation can be decomposed into two reflections.
<u>Proof</u>: In the diagram, the reflection lines *e* and *f* are parallel. Assume point *A* is on the opposite side of *e* from *f* as shown. *A* reflects to *A′* in *e*; *A′* reflects to *A″* in *f*. By the definition of reflection, $AA' \perp e$, $A'A'' \perp f$, and the equal segments are labeled *x* and *y*.



Since line j, which contains *A* and *A′*, is perpendicular to *e*, it is also perpendicular to *f* by Theorem 15. So, *A*, *A′*, and *A″* are collinear. Therefore, *A* translates to *A″* by the vector shown, which is perpendicular to *e* and *f*. Its length, $2(x + y)$, is twice the distance $x + y$ between *e* and *f*.
If *A* is between *e* and *f*, the diagram looks like this:

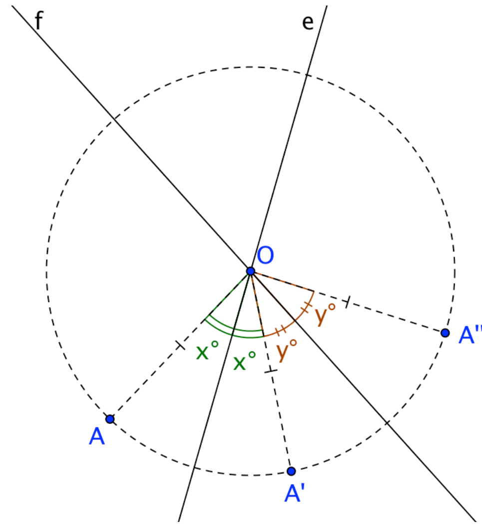


As before, the distance between *A* and *e* is *x* and the distance between *A′* and *f* is *y*. Now the distance between *e* and *f* is $y - x$ and the distance between *A* and *A″* is $2y - 2x$, but the argument is essentially the same.
This argument also applies in the third case, where *A* is on the opposite side of *f* from *e*. The diagram for this case is below.



For a dynamic view of this with a quadrilateral as pre-image, see Theorem 3 on this web page:
http://www.mathedpage.org/transformations/isometries/four/index.html#two .

17. The composition of two reflections in intersecting lines is a rotation around their point of intersection. The angle of rotation is twice the directed angle between the lines going from the first reflection line to the second (either clockwise or counterclockwise). This implies that any rotation can be decomposed into two reflections.
Proof: The argument is like Theorem 16. In the diagram, the reflection lines e and f intersect at O and the resulting rotation is counterclockwise. A reflects to A' in e; A' reflects to A'' in f. Since reflection preserves angles and the reflection line is fixed, the equal angles are labeled x and y. By the definition of rotation, $OA = OA' = OA''$. One angle between e and f is $x + y$, and the counterclockwise rotation angle from A to A'' is $2x + 2y = 2(x + y)$, twice the angle between the lines. The other angle between e and f is $180 - (x + y)$ and the clockwise rotation angle is $360 - 2x - 2y = 360 - 2(x + y) = 2[180 - (x + y)]$, also twice the angle between the lines.



For a dynamic view of this with a quadrilateral as pre-image, see Theorem 4 on this web page:
http://www.mathedpage.org/transformations/isometries/four/index.html#two .

18. * Rotation, and translation preserve segment length, angle measure. collinearity, and betweenness.
Proof: We assumed that reflection preserves segment length and angle measure. Theorem 2 shows that reflection preserves collinearity and betweenness. Since rotation and translation are compositions of two reflections, they preserve these also.

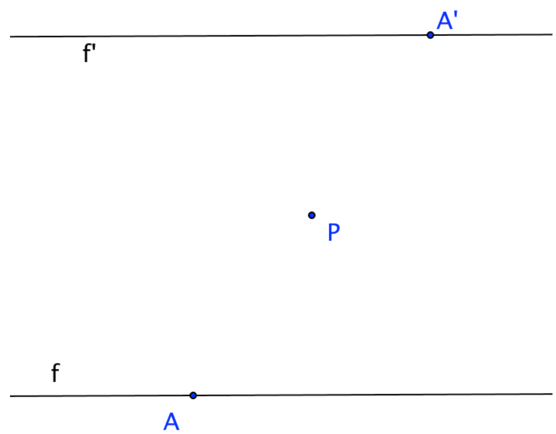
19. * If $A \rightarrow A'$ and $B \rightarrow B'$ under a rotation or translation, segment AB must map onto segment $A'B'$.
Proof: Both rotation and translation can be decomposed into two reflections. Since the theorem is true for reflections (Theorem 3), it must be true for rotation and translation.

20. * Reflections, rotations, and translations map rays onto rays and lines onto lines.
Proof: Both rotation and translation can be decomposed into two reflections. Since the theorem is true for reflections (Theorem 4), it must be true for rotation and translation.

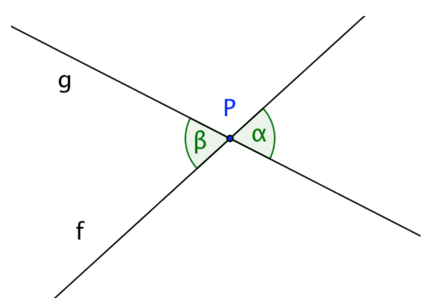
21. * Given two intersecting lines, there are two reflections mapping one to the other.
Proof: The lines that bisect each of two adjacent angles formed at the intersection point are the lines of reflection because reflection preserves angle measure.
Note: The related theorem for distinct parallel lines is Theorem 10 of Chapter 3.

Half-Turns, Vertical Angles, Transversals, and Translations

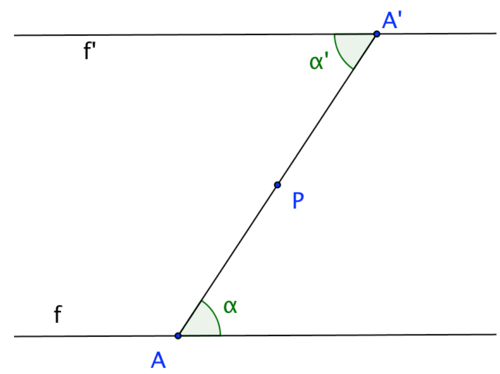
22. * If $A' = B$ under a half-turn, then $B' = A$.
    Proof: 180° rotation clockwise is equivalent to 180° rotation counter-clockwise. So, if a half-turn takes A to B, its inverse (itself) will take B to A.
23. * The image of segment AB under a half-turn around its midpoint is the segment BA. That is, $A' = B$ and $B' = A$.
    Proof: If M is the midpoint of segment AB, $\angle AMB = 180°$ and $MA = MB$. By the definition of rotation, $A' = B$ and $B' = A$ under a half-turn around M. Since the image of a segment is a segment, AB is invariant under the half-turn.
24. * A line is its own image under a half-turn around a point on the line.
    Proof: If O is the center of rotation and P is a point on the line, $\angle POP' = 180°$ under the half-turn around O. Therefore, P' is also on the line. Under the half-turn, $(P')' = P$, so every point on the line is the image of another point.
25. * The image of a line under a half-turn is parallel to the pre-image.
    Proof: Theorem 23 proves this if the center of rotation is on the line, so suppose that it isn't. f' is the image of f and A' is the image of A under a half-turn H around P. By Theorem 21, H will take A' back to A. Since this is true for all points A on line f, H will take f' back to f. Suppose f' is not parallel to f, so that f' and f intersect at Q. Q must be distinct from P, because it lies on f. Let $Q' = H(Q)$. Q' is an intersection point of f and f' that is distinct from Q. Since f' and f intersect in two distinct points, $f' = f$. But if A' is on f, then P is also, contradicting our assumption.

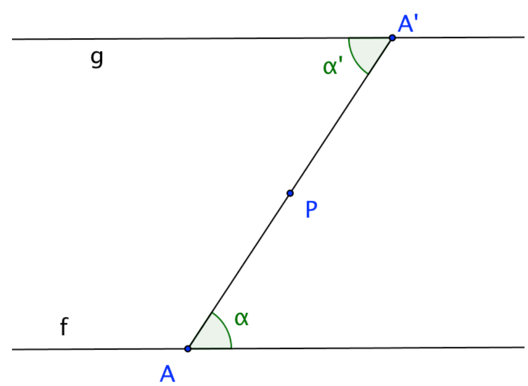


26. * When two lines intersect, the vertical angles are equal.
    Proof: Rotate angle α 180° around P. α' = α because rotation preserves angles. But lines f and g are their own images by Theorem 23. So α' = β and therefore α = β.

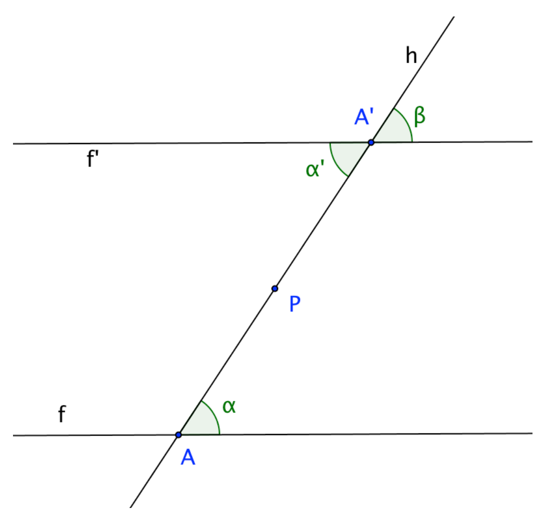


27. * If two distinct lines are cut by a transversal, they are parallel if and only if the alternate interior angles are equal.
<u>Proof</u>: Suppose the line we called f' is parallel to f and the transversal intersects f at A and f' at a point we'll call A'. Let P be the midpoint of segment AA'. Let H be the half-turn around point P. By Theorem 22, A' is the image of A under H. By Theorem 24, f' is the image of f. Since P is fixed under H, segment PA' is the image of segment PA. Therefore, angle α' is the image of angle α, and because rotation preserves angle measure, $\alpha' = \alpha$.
Conversely, suppose in the second diagram that that f and g are two distinct lines, and that P is the midpoint of transversal segment AA'. Furthermore, suppose that the alternate interior angles labeled α and α' are equal. Since rotation preserves angles and segment PA' is one side of $H(\alpha)$, $H(f) = g$. By Theorem 24, g is parallel to f.





28. * If two distinct lines are cut by a transversal, they are parallel if and only if the corresponding angles are equal.
<u>Proof</u>: By Theorem 26, f' is parallel to f if and only if $\alpha' = \alpha$. By Theorem 25, $\beta = \alpha'$. Therefore, f' is parallel to f if and only if $\beta = \alpha$.



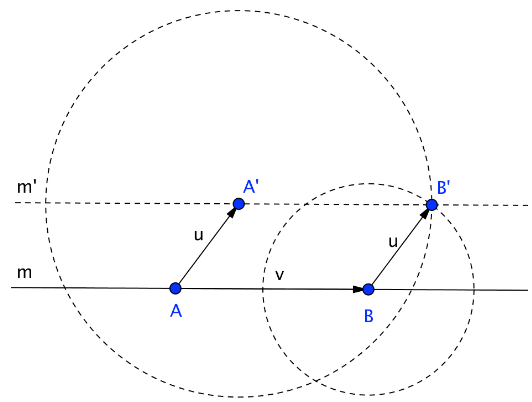
29. * The composition of translations is commutative.

Notation: $\mathrm{T}_u(B)$ is the image of B under a translation with vector u. $\mathrm{T}_u(\overleftrightarrow{AB})$ is the image of line AB under a translation with vector u, and similarly with segments and vectors.

Proof: Let u and v be the two translation vectors. If the vectors are parallel, take representatives on the same line. Mark this line with numbers to make it a number line. The result follows from commutativity of addition.

If they are not parallel, let $\mathrm{T}_u(A) = A'$, $\mathrm{T}_u(B) = B'$ and $\mathrm{T}_v(A) = B$. We need to show that $\mathrm{T}_v(A') = B'$.

Let $\mathrm{T}_v(A') = C$. By the definition of vector, $A'C = AB$ (the length of v).
$A'B' = AB$ also, since $A'B' = \mathrm{T}_u(AB)$ and translation preserves distance. Since $A'B' = A'C$, both B' and C are on a circle centered at A', with radius AB.
Likewise, $BB' = AA'$ (the length of u), and $BC = AA'$ since $BC = \mathrm{T}_v(AA')$. Therefore, both B' and C are on a circle centered at B, with radius AA'.



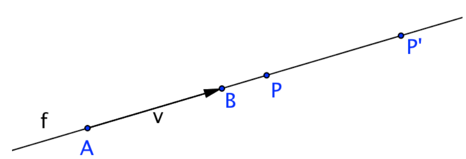
Thus, B' and C are both at the intersection of the two circles. Let m be the line through A and B. Then $m' = \mathrm{T}_u(m)$ is the line through A' and B'. B' must be on the same side of m as A', because the other intersection is $\mathrm{T}_{-u}(\mathrm{B})$. C must be on the same side of m as A' because $\mathrm{T}_v(A') = C$ and v is the direction of m. Therefore B' and C are the same intersection point, so $\mathrm{T}_\mathrm{u}(\mathrm{T}_\mathrm{v}(A) = \mathrm{T}_\mathrm{u}(B) = B'$ and $\mathrm{T}_\mathrm{v}(\mathrm{T}_\mathrm{u}(A) = \mathrm{T}_\mathrm{v}(A') = B'$.

Pedagogical note: It is essential to make clear that in general, composition is not commutative. For example, students could experiment on interactive geometry software.

30. * The translation image of a line is parallel to the line.

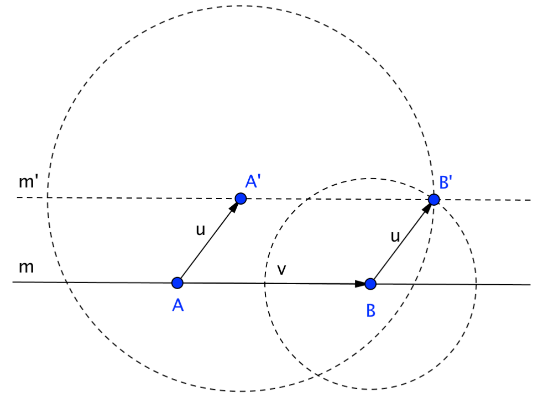
Proof:

Part 1: The translation image of a line is the line itself if the vector is parallel to the line.



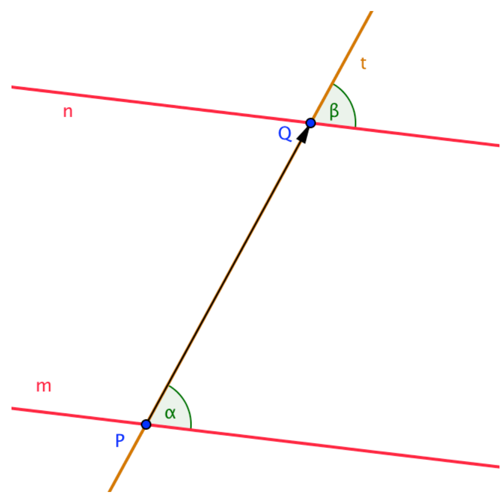
Proof: Let $\vec{v} = \overrightarrow{AB}$ be a representative of the translation vector with A on line f. If P is an arbitrary point on line f, $\overrightarrow{PP'} = \vec{v}$ by the definition of translation. P' lies on f because $\vec{v}$ moves points in the direction of the line. To locate the pre-image of any point on the line under $\vec{v}$, apply $-\vec{v}$ to it. So, the image of the line is the entire line.

Part 2: The translation image of a line is a distinct line parallel to its pre-image if the vector is not parallel to the line.



Proof: We can use the same diagram and setup as Theorem 28. Consider a line *m* and a vector *u* not parallel to *m.* Choose two arbitrary points *A* and *B* on *m*. Let $T_u(A) = A'$, $T_u(B) = B'$, and $\overrightarrow{AB} = v$. We have already shown that $T_v(A') = B'$, and *u* does not have the same direction as *v*, so the two representations $\overrightarrow{AB}$ and $\overrightarrow{AB}$ of *v* are parallel and not collinear. We have shown that the image of any segment on *m* is a parallel segment on *m'*. Therefore, $m' = T_u(m)$ is parallel to *m*.

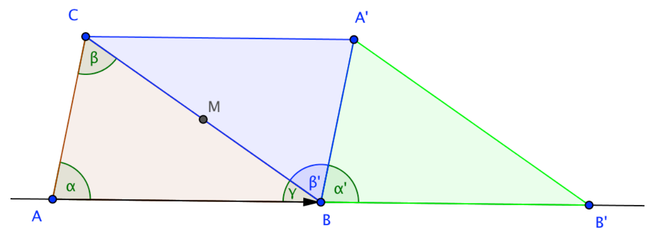
31. * Any representative of a vector can be superimposed on any other by a translation. Proof: If *u* and *v* are two representatives of the same vector, translate the initial point of *u* to the initial point of *v* by translation T. Since *u* and *v* are parallel, since translation maps any line into a parallel line, and since *u* and *v* have the same length and point in the same direction, T(*u*) = *v*.
32. * If two distinct lines are cut by a transversal, an angle on one line is a translation image of an angle on the other if and only if the lines are parallel.
a) If two parallel lines are cut by a transversal, an angle on one parallel is the translation image of an angle on the other.



Proof: Given distinct parallel lines m and n and transversal t shown in the diagram, translate m by vector $\overrightarrow{PQ}$. *m'* contains *Q* and is parallel to *m* by Theorem 29, Part 2. By the parallel postulate, *m'* = *n*. *t'* = *t* by Theorem 29, Part 1. Therefore α' = β. Since translation preserves angles, α = β.
b) If two lines are cut by a transversal, and if an angle on one line is the translation image of an angle on the other, then the lines are parallel.
Proof: Suppose angle β is a translation image of angle α, so that α' = β. Because intersection must map to intersection, $\overrightarrow{PQ}$ is the translation vector and t is a transversal because it is common to both angles. Since α' = β, *m'* = *n*. By Theorem 29, Part 2, *m* || *n*.

Sum of Angles

33. The sum of the angles of a triangle is 180˚. An exterior angle of a triangle is equal to the sum of the remote interior angles. Therefore it is greater than either one.
    Proof: Rotate ΔABC 180˚ around M, the midpoint of segment BC. By Theorem 22, B' = C and C' = B. β' = β because rotation preserves angles and $BA' \parallel AC$ by Theorem 26. Now translate ΔABC by vector $\overrightarrow{AB}$. A, B and B' are collinear by Theorem 29 Part 1, and C' = A' for two reasons:



    the image of AC is a line through B parallel to AC by Theorem 29 Part 2, and BA' = AC because translation preserves distance. Because $\angle ABB'$ = 180˚, α' + β' + γ = α + β + γ = 180˚. Also, from the diagram, exterior $\angle CBB'$ = α + β and is greater than either α or β.
    Pedagogical Note: An informal version of this proof can be discussed with students after asking them to find a tessellation based on a scalene triangle tile.)
34. The sum of the interior angles of a quadrilateral is 360˚. (In a concave quadrilateral, one interior angle will be greater than 180˚. Such an angle is called a reflex angle.)
    Proof: Draw a diagonal from the vertex of a reflex angle if there is one, arbitrarily otherwise. The diagonal divides the quadrilateral into two triangles, the sum of whose angles is 180˚. The angles of both triangles, together, make up the interior angles of the quadrilateral.

Chapter 2: Symmetry Definitions and Properties - Triangles and Quadrilaterals

We define special triangles and quadrilaterals in terms of their symmetries. We begin each section with a symmetry definition, then use it to deduce the figure's properties. The proofs are generally easier than the traditional approach.

Chapter 3 deals with the converse: we supply conditions and prove that the figure has the defining symmetry. This is a more difficult task. By the end of Chapter 3, we will have shown that the traditional definitions are equivalent to the symmetry definitions.

The general trapezoid is a special case, because it has no symmetry. We can use a transformation to define it, but not an isometry. For that reason, we postpone theorems about a general trapezoid until Chapter 4: *Basics of Dilation and Similarity*.

Our definitions are inclusive, which is logical, and moreover consistent with the behavior of dynamic geometry software. For example, you can drag a vertex of a dynamic isosceles trapezoid to make it a rectangle, so we consider a rectangle as a special isosceles trapezoid. Similarly, an equilateral triangle is a special isosceles triangle, a rhombus is a special kite, and so on.

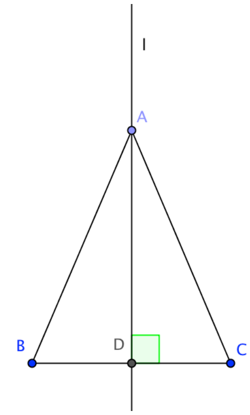
<u>Pedagogical Note:</u> Students most likely are already familiar with the traditional definitions of special triangles and quadrilaterals. This chapter can be preceded with the Symmetric Polygons activity, where students are given symmetry definitions, and asked to recognize which polygon can be defined that way. [insert link here]

1. * The image of a vertex in a line of symmetry is also a vertex.
   <u>Proof</u>: A vertex is the common endpoint of two sides. Because collinearity is preserved, sides must map onto sides. So, the image of a vertex must also lie on two sides. A point on two sides is a vertex, so it must also be a vertex.

2. <u>Isosceles Triangle</u>: A triangle with at least one line of symmetry.

   The etymology of "isosceles", of course, is "equal legs". In the scheme we propose, this is no longer the definition; it is a property that must be proved. (See property (b) below.)

   Properties:
   a. One vertex lies on the line of symmetry and the other two are each other's reflections.
      <u>Proof</u>: Because there is an odd number of vertices, one of them must lie on the line of symmetry.



b. An isosceles triangle has two equal sides and two equal angles.
<u>Proof</u>: Reflection preserves side lengths and angle measure. If vertex A is on the line of symmetry, then $AB = AC$ and $\angle B = \angle C$.

c. The perpendicular bisector of the third side of an isosceles triangle bisects an angle of the triangle, so the line of symmetry is an altitude, a median, and a perpendicular bisector.
<u>Proof</u>: By the definition of reflection, the line of symmetry l is the perpendicular bisector of BC. It also must pass through A. Since reflection preserves angles,
$\angle D\mathrm{AB} = \angle DAC$, so l bisects $\angle BAC$. l is clearly an altitude, median, and perpendicular bisector.

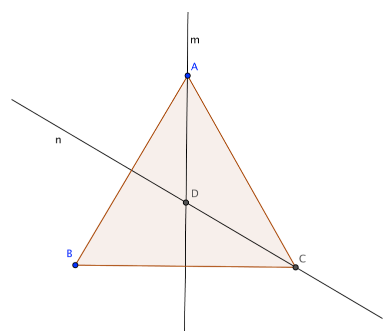
3. <u>Equilateral Triangle</u>: A triangle with at least two lines of symmetry.

Other (equivalent) definitions are possible. We prefer this one, as it is economical, and facilitates the proof of properties.

Note that once again, the etymology does not correspond to the definition; we must prove all sides are equal. (See property (b) below.)

Properties:

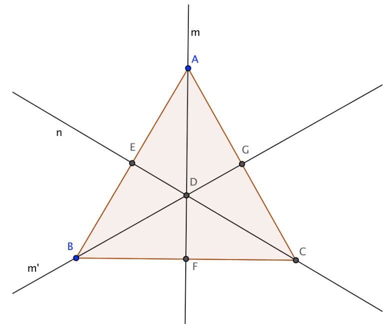
a. An equilateral triangle has 3-fold rotational symmetry.
<u>Proof</u>: Let m and n be the symmetry lines through A and C respectively. The composition of reflections in m, then n maps the triangle onto itself and is a rotation around their intersection point D. Call this rotation r. r maps A onto B, B onto C, and C onto A. Repeating this rotation three times gives the identity transformation, so the triangle has 3-fold rotational symmetry around the intersection point of its two lines of symmetry.



b. All sides of an equilateral triangle are equal and each angle is 60˚.
<u>Proof</u>: Rotation preserves side lengths and angle measure. Since the sum of the angles in a triangle is 180˚, each angle is 60˚.

c. An equilateral triangle has three concurrent lines of symmetry.
<u>Proof</u>: r maps A to B and D to itself, so m', the image of m under r, passes through B and D. Since m perpendicularly bisects BC, m' must

perpendicularly bisect CA because rotation preserves segment length and angle measure. Therefore m' is a third line of symmetry of ΔABC.



d. Each line of symmetry of an equilateral triangle is an altitude, a median, and a perpendicular bisector.
<u>Proof</u>: The triangle is isosceles in three different ways.

The next figure, a parallelogram, is defined using rotational symmetry, so we need a few comments about vocabulary. If a figure coincides with itself *n* times in one complete turn about a specified point (often called its center), we say the figure has *n*-fold rotational symmetry. For example, a regular pentagon has 5-fold rotational symmetry. We also say that a regular pentagon has 72˚ rotational symmetry, because a rotation of 72˚ around its center will map it onto itself. We could make the same statement using any integer multiple of 72˚, but it's more descriptive to use the smallest positive angle. In general, the smallest rotation angle for a figure with n-fold rotational symmetry is $\frac{360^\circ}{n}$.

4. <u>Parallelogram:</u> A quadrilateral with 2-fold (180˚) rotational symmetry.
<u>Note</u>: This figure and a general trapezoid are the only special quadrilaterals whose definitions do not involve line symmetry.

Properties:

a. * The image of a vertex under the symmetry rotation is an opposite vertex.
<u>Proof</u>: Let *r* be the 2-fold rotation. *r* followed by *r* ($r \circ r$) is a 360° rotation, i.e. the identity. As with triangles, the image of a vertex under *r* must be a vertex. Its image under $r \circ r$ must be itself. If the image were a consecutive vertex, then the image under $r \circ r$ would be the next consecutive vertex (i.e. the opposite vertex), not the original. Therefore, the image is the opposite vertex.

b. The center of the 2-fold rotation is the common midpoint of the diagonals.
<u>Proof</u>: A diagonal must rotate into itself because its endpoints switch. So, a diagonal must contain the center of rotation. The distance from the center to one endpoint must equal the distance to the other because rotation preserves distance. Therefore, the center must be the midpoint of either diagonal, which implies it is the common midpoint of both.

c. The opposite sides of a parallelogram are parallel.
<u>Proof</u>: The image of a line under a half-turn around a point not on the line is a parallel line.

d. Consecutive angles of a parallelogram are supplementary
Proof: This is a property of parallel lines cut by a transversal.

e. * The image of a side under *r* is an opposite side. The image of an angle under *r* is an opposite angle.
Proof: The image can't be a consecutive side because then it wouldn't be parallel to the pre-image. The image of an angle can't be a consecutive angle because then one image side wouldn't be parallel to its pre-image.

f. The opposite sides and opposite angles of a parallelogram are equal.
Proof: Rotation preserves segment length and angle measure.

5. Kite: A quadrilateral with at least one line of symmetry through opposite vertices. (It is possible to omit "opposite" from the definition and prove that if a line of symmetry passes through vertices, they must be opposite. For most students, this sort of subtlety would be counterproductive. On the other hand, it would make for an interesting discussion.)

Properties:

a. A kite has two disjoint pairs of consecutive equal sides and one pair of equal opposite angles. (We need to say "disjoint," because the pairs can't have a common side.)
Proof: Reflection in the line of symmetry preserves segment length and angle measure.

b. The line of symmetry of a kite bisects a pair of opposite angles.
Proof: Reflection preserves angle measure.

c. The diagonal of a kite that lies on the line of symmetry perpendicularly bisects the other diagonal.
Proof: The symmetry line perpendicularly bisects the segment joining the vertices not on the line because they reflect into each other.

6. Isosceles Trapezoid: A quadrilateral with a line of symmetry though interior points of opposite sides. If there is only one such line of symmetry, these sides are called *bases*. In this case, the other two sides are called *legs*.

Properties:

a. Two vertices of an isosceles trapezoid are on one side of the symmetry line and two are on the other.
Proof: Since reflection maps vertices to vertices, the four vertices must be evenly split on either side of the symmetry line.

b. The symmetry line of an isosceles trapezoid is the perpendicular bisector of the bases.
Proof: One endpoint of each of these sides must reflect into the other. A reflection line perpendicularly bisects the segment joining pre-image and image points if these points are not on the reflection line.

c. The bases of an isosceles trapezoid are parallel.
Proof: They are both perpendicular to the symmetry line. Two distinct lines perpendicular to the same line are parallel.

d. The legs of an isosceles trapezoid are equal.
Proof: Reflection preserves segment length.

e. Two consecutive angles of an isosceles triangle on the same base are equal.
Proof: Reflection preserves angle measure.

f. The diagonals of an isosceles trapezoid are equal.
Proof: One diagonal reflects to the other. Reflection preserves segment length.

g. The intersection point of the equal diagonals of an isosceles trapezoid lies on the symmetry line.
Proof: The point where one diagonal intersects the symmetry line must be invariant under reflection in the symmetry line because it lies on it. Therefore, it also lies on the other diagonal.

h. The intersection point of the diagonals of an isosceles trapezoid divides each diagonal into equal subsections.
Proof: The subsections of one diagonal determined by the intersection point reflect onto the subsections of the other. These subsections are equal because reflection preserves segment length.

7. Rhombus: A quadrilateral with two lines of symmetry passing through opposite vertices. (So, a rhombus is a kite in two different ways.)

Properties:

a. A rhombus has all sides equal and two pairs of equal opposite angles.
Proof: A kite has two disjoint pairs of consecutive equal sides and one pair of equal opposite angles. The result follows because a rhombus is a kite in two different ways (i.e. both diagonals are lines of symmetry).

b. Each diagonal of a rhombus bisects its angles.
Proof: Each line of symmetry bisects a pair of opposite angles (property of kites).

c. The diagonals of a rhombus perpendicularly bisect each other.
Proof: The diagonal of a kite that lies on the line of symmetry perpendicularly bisects the other diagonal. For a rhombus, each diagonal has this property.

d. A rhombus is a special parallelogram.
Proof: Since a rhombus has two perpendicular lines of symmetry, the composition of reflection in these lines is a 180˚ rotation around their point of intersection. (The composition of two reflections is a rotation around their point of intersection through twice the angle between the reflection lines.) Since each reflection maps the rhombus onto itself, their composition does also.

e. The opposite sides of a rhombus are parallel.
Proof: Since a rhombus is a parallelogram, the opposite sides are parallel.

8. Rectangle: A quadrilateral with two lines of symmetry passing through interior points of the opposite sides. (So, a rectangle is an isosceles trapezoid in two different ways.)

Properties:

a. The symmetry lines of a rectangle perpendicularly bisect the opposite sides.
Proof: A rectangle is an isosceles trapezoid in two different ways.

b. A rectangle is equiangular.
Proof: Two consecutive angles of an isosceles trapezoid that share a base are equal. Both pairs of opposite sides are bases because of the two different ways, so any two consecutive angles share a base.

c. All angles of a rectangle are right angles.
Proof: The sum of the interior angles of any quadrilateral is 360˚ and $360 \div 4 = 90$.

d. The symmetry lines of a rectangle are perpendicular.
Proof: The lines divide the rectangle into four quadrilaterals. Each has three right angles: one is an angle of the rectangle and the other two are formed by a side and a symmetry line, which are perpendicular. Since the sum of the angles of a quadrilateral is 360˚, the fourth angle at the intersection of the symmetry lines must also be a right angle.

e. A rectangle has 2-fold rotational symmetry, so it is a special parallelogram.
Proof: Reflecting a rectangle in one line of symmetry followed by the other maps the rectangle onto itself and is equivalent to a 180˚ rotation because the symmetry lines meet at right angles. Therefore, a rectangle has 2-fold rotational symmetry around the intersection of the symmetry lines.

f. The opposite sides of a rectangle are parallel and equal.
Proof: These are properties of a parallelogram. A rectangle is a special parallelogram.

g. The diagonals of a rectangle are equal.
Proof: This is a property of an isosceles trapezoid. A rectangle is a special isosceles trapezoid.

h. The diagonals of a rectangle bisect each other.
Proof: This is a property of a parallelogram. A rectangle is a special parallelogram.

i. The diagonals of a rectangle and the lines of symmetry are all concurrent.
Proof: The intersection point of the equal diagonals of an isosceles trapezoid lies on the symmetry line. For a rectangle, the intersection point lies on both symmetry lines, so it is their intersection.

9. Square: A quadrilateral with four lines of symmetry: two diagonals and two lines passing through interior points of opposite sides.

Properties:

a. A square is a special rectangle, rhombus, kite, and isosceles trapezoid, so it inherits all the properties of these quadrilaterals.
Proof: True, by the definition of a square.

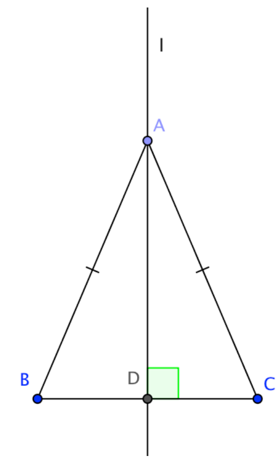
b. If a square and all four symmetry lines are drawn, all the acute angles are 45˚.
Proof: The diagonals bisect the interior right angles because a square is a rhombus. All eight right triangles formed have a right angle where the symmetry lines intersect the sides and a 45˚ angle where they intersect the vertices. Since the sum of the angles of a triangle is 180˚, the remaining angles at the center must all be 45˚.

c. A symmetry line through sides and a symmetry line through vertices form a 45˚ angle.
Proof: An immediate consequence of the result just above.

d. A square has 4-fold rotational symmetry.
Proof: Reflecting a square in a line of symmetry through the sides followed by a line of symmetry through the vertices maps the square onto itself. It is equivalent to a 90˚ rotation because these symmetry lines meet at a 45˚angle. Therefore, a square has 4-fold rotational symmetry around its center (the intersection point of the lines of symmetry).

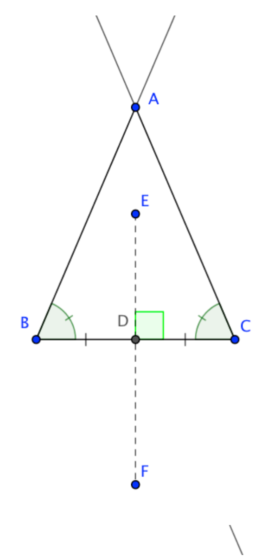
Chapter 3: Proving Triangles and Quadrilaterals Satisfy Symmetry Definitions

1. Isosceles Triangle: A triangle with one line of symmetry.
   a. If a triangle has two equal sides, it is isosceles.
   Proof: Let AB and AC be the equal sides. A must lie on the perpendicular bisector l of BC because it is equidistant from B and C. By the definition of reflection, $B' = C$ under reflection in l. $A' = A$ because it lies on l. Therefore, l is a line of symmetry for ΔABC.

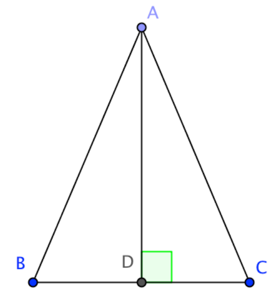


   b. If a triangle has two equal angles, it is isosceles.
   Proof 1: By Theorem 21 of Chapter 1, the rays AB and AC are reflections of each other in the angle bisector AD of $\angle BAC$. Two angles of ΔBAD and ΔCAD are equal, so the third angles ($\angle BDA$ and $\angle CDA$) must be equal. Since they are supplementary, they are both right angles. It follows that the reflection of B in ray AD must be C. Since A is its own reflection in line AD, AD is a line of symmetry for the triangle.

   Proof 2: Draw segment EF, which perpendicularly bisects side BC. B and C interchange under this reflection. Because reflection preserves angles and $\angle B = \angle C$, ray BA and ray CA interchange also. A is on both rays, so A' must be on both rays also. But A is the only point on both rays, so $A' = A$ under this reflection. Since A is a fixed point, it lies on the reflection line. Therefore, line EF is a line of symmetry for the triangle.



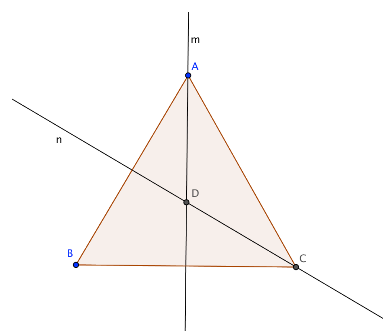
c. If an angle bisector of a triangle is also an altitude, the triangle is isosceles.
Proof: Let *l*, the bisector of $\angle BAC$ be perpendicular to side BC at D, so that $\angle DAB = \angle DAC$ and $\angle ADB = \angle ADC$. Two angles of ΔBAD and ΔCAD are equal, so the third angles ($\angle B$ and $\angle C$) must be equal. By Theorem 1b above, the triangle is isosceles.



d. If an altitude of a triangle is also a median, the triangle is isosceles.
Proof: Since altitude AD is also a median, AD is the perpendicular bisector of BC. Since any point on the perpendicular bisector of a segment is equidistant from the endpoints, $AB = AC$. By Theorem 1a above, the triangle is isosceles.

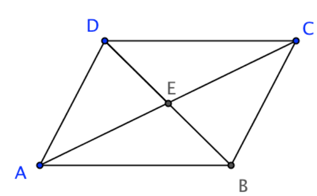
e. If an angle bisector of a triangle is also a median, the triangle is isosceles.
Proof: Postponed until the end of the rhombus section (after Theorem 6e), because a rhombus is constructed in the proof.

2. Equilateral Triangle: A triangle with two lines of symmetry.
   a. If a triangle has all *sides* equal, then it's an equilateral triangle.
   Proof: In ΔABC, $AB = BC = CA$. Since $AB = AC$, the triangle is isosceles with symmetry line m. Since $CA = CB$, the triangle is isosceles with symmetry line n. Since it has two lines of symmetry, it is equilateral.

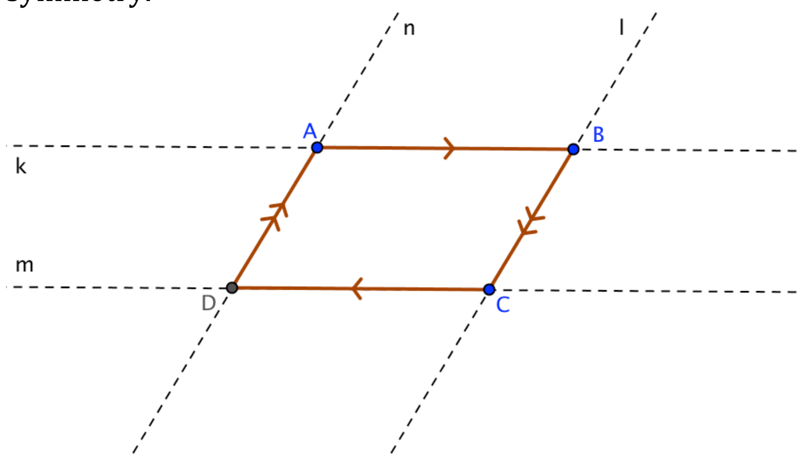


   b. If a triangle has all *angles* equal, then it's an equilateral triangle.
   Proof: The argument is virtually identical to the previous one, but uses Theorem 1b instead of 1a.

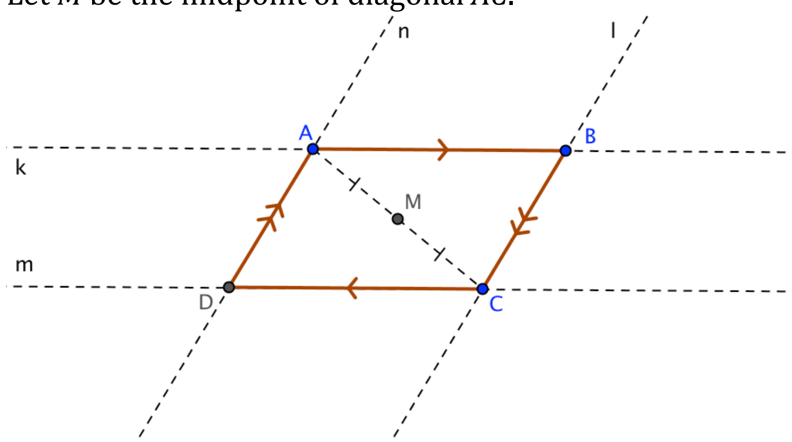
3. Parallelogram: A quadrilateral with 2-fold rotational symmetry.
   a. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
   Proof: Rotate quadrilateral $ABCD$ 180˚ around point E, the intersection of the diagonals. Since the rotation is 180˚, B' lies on ray ED. Since rotation preserves distance, $B' = D$. Similarly, $A' = C$. Similarly, $C' = A$ and $D' = B$. Because rotation maps segments to segments, each side of $ABCD$ maps to the opposite side. Therefore, quadrilateral ABCD has 2-fold rotational symmetry. By definition, $ABCD$ is a parallelogram.



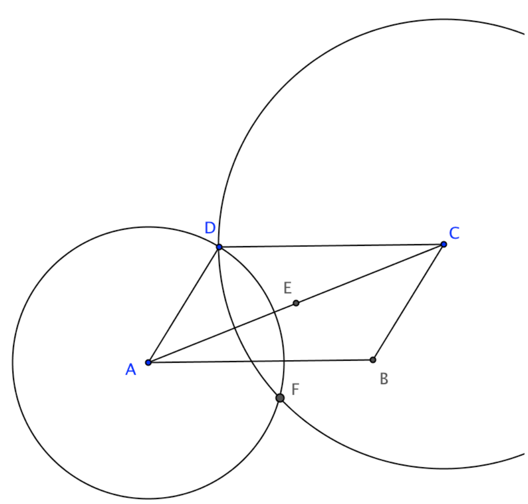
b. If opposite sides of a quadrilateral are parallel, the quadrilateral is a parallelogram.
Proof: Given quadrilateral *ABCD* as in this figure, with sides extended to lines *k*, *l*, *m*, and *n*. We would like to prove that it has 2-fold (half-turn) symmetry.



Let *M* be the midpoint of diagonal *AC*.

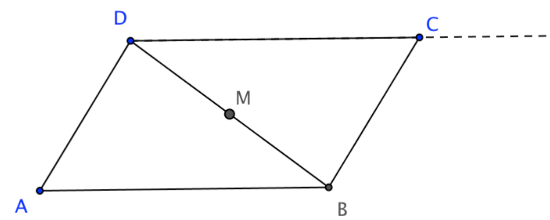


Consider *H*, the half-turn with center *M*. Since *M* is the midpoint of segment *AC*, *A'* = *C* and *C'* = *A* under *H*. Because *M* is on neither *k* nor *l*, their image lines are parallel to their pre-images. Because of the parallel postulate, there is only one parallel to *k* through *C* and one parallel to *l* through *A*. Therefore, *k'* = *m*, and *l'* = *n*. *B* is the intersection of lines *k* and *l*, and therefore its image is the intersection of lines *m* and *n*, which is *D*. Since *A'* = *C* and *B'* = *D*, *ABCD* has 2-fold rotational symmetry. By definition, *ABCD* is a parallelogram.



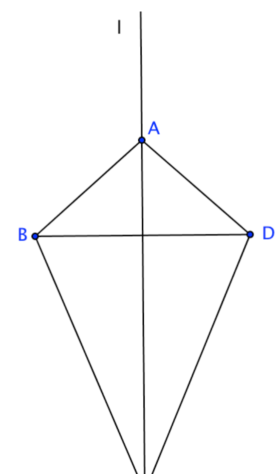
c. If opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.
<u>Proof</u>: In quadrilateral $ABCD$, draw diagonal AC and its midpoint E. Under a half turn around E, $A' = C$ and $C' = A$. Since $CB = AD$, B' lies on circle A with radius AD. Since AB = CD, B' lies on circle C with radius CD. These circles intersect at D and F. But F is on the same side of line AC as B, so $B' \neq F$. Therefore $B' = D$ and $ABCD$ has 2-fold symmetry around E. By definition, $ABCD$ is a parallelogram.

d. If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.
<u>Proof</u>: In quadrilateral $ABCD$, suppose AB is equal and parallel to DC. Draw diagonal BD and its midpoint M. Under a half-turn around M, $B' = D$ and $D' = B$. The image of ray BA

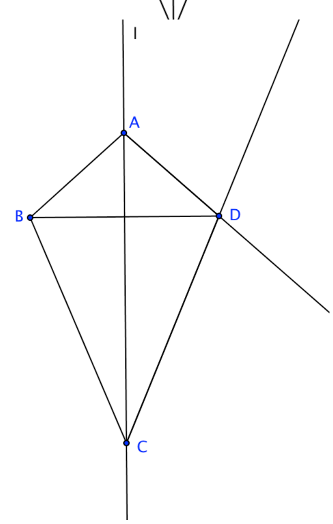


is parallel to AB, so it must coincide with ray DC. Because $AB = DC$, that means that $A' = C$, and therefore $C' = A$. Hence $ABCD$ is a parallelogram.

4. <u>Kite:</u> A quadrilateral with one line of symmetry through opposite vertices.
   a. If two disjoint pairs of consecutive sides of a quadrilateral are equal, the quadrilateral is a kite.
   <u>Proof</u>: In quadrilateral $ABCD$, suppose $AB = AD$ and $CB = CD$. Since A and C are both equidistant from B and D, they lie in the perpendicular bisector of diagonal BD. Therefore, l is the perpendicular bisector of diagonal BD. Under reflection in l, $B' = D$ and $D' = B$. Because A and C both lie on l, $A' = A$ and $C' = C$. l is therefore a line of symmetry and $ABCD$ is a kite.



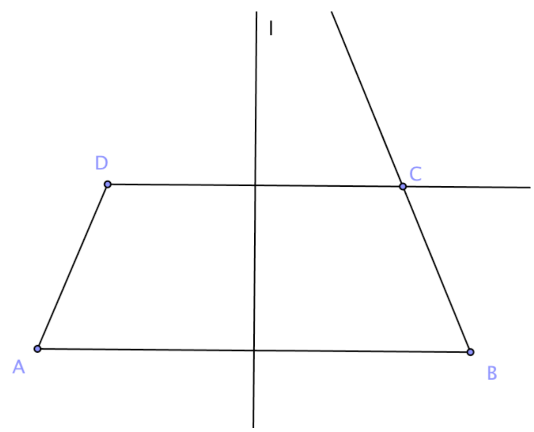
   b. If a diagonal of a quadrilateral bisects a pair of opposite angles, the quadrilateral is a kite.
   <u>Proof</u>: Label as l the line through diagonal AC that bisects $\angle BAD$ and $\angle BCD$. Consider reflection in l. Since A and C are on l, $A' = A$ and $C' = C$. Since $\angle BAC = \angle DAC$, B' lies on ray AD. Since $\angle BCA = \angle DCA$, B' lies on ray CD. Because these rays intersect at D, $B' = D$, which implies that $D' = B$. Therefore, l is a line of symmetry and $ABCD$ is a kite.



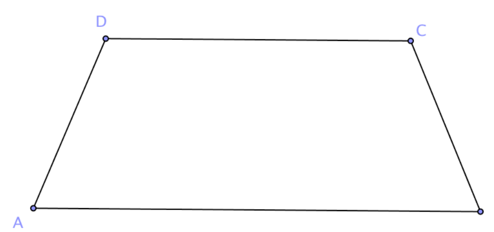
c. If one diagonal of a quadrilateral perpendicularly bisects the other, the quadrilateral is a kite.
Proof: In quadrilateral $ABCD$, diagonal AC perpendicularly bisects diagonal BD. Let l be the line through A and C. Reflect $ABCD$ in l. Since A and C lie on l, $A' = A$ and $C' = C$. By the definition of reflection, $B' = D$ and $D' = B$. Therefore, l is a line of symmetry and $ABCD$ is a kite.

5. Isosceles Trapezoid: A quadrilateral with a line of symmetry though interior points of opposite sides.

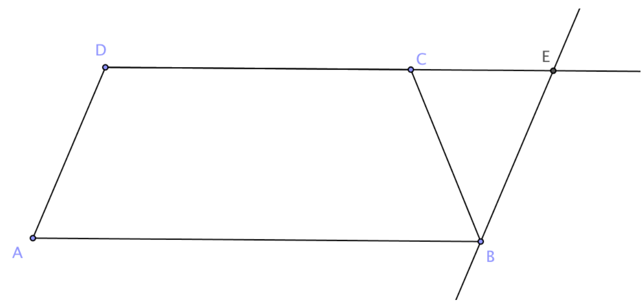
a. If one pair of opposite sides of a quadrilateral are parallel and a pair of consecutive angles on one of these sides are equal, the quadrilateral is an isosceles trapezoid.
Proof: In quadrilateral $ABCD$, $DC \parallel AB$ and $\angle A = \angle B$. Let l be the perpendicular bisector of AB. Under reflection in l, $A' = B$ and $B' = A$. Since $DC \parallel AB$, $DC \perp l$. Therefore, D' lies on ray DC. Because $A' = B$, ray AB maps to ray BA, $\angle A = \angle B$, and reflection preserves angles, D' lies on ray BC. These rays intersect at C, so $D' = C$. Thus, l is a line of symmetry for $ABCD$ and $ABCD$ is an isosceles trapezoid.



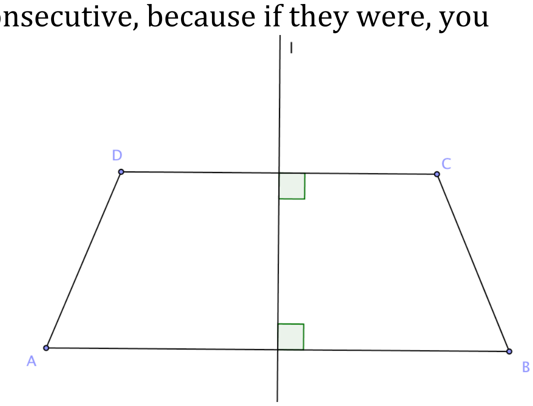
b. If two disjoint pairs of consecutive angles of a quadrilateral are equal, the quadrilateral is an isosceles trapezoid.
Proof: In quadrilateral $ABCD$, $\angle B = \angle A$ and $\angle C = \angle D$. Because $\angle A + \angle B + \angle C + \angle D = 360^\circ$, $2\angle A + 2\angle D = 360^\circ$. Dividing both sides by 2 gives $\angle A + \angle D = 180^\circ$, so $DC \parallel AB$. By Theorem 5a, $ABCD$ is an isosceles trapezoid.



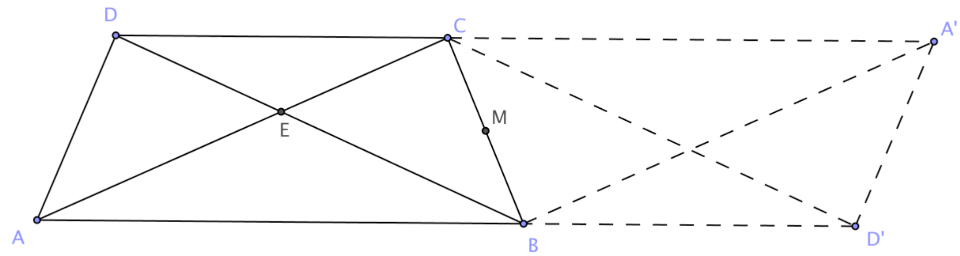
c. If two opposite sides of a quadrilateral are parallel and if the other two sides are equal but not parallel, then the quadrilateral is an isosceles trapezoid.
Proof: In quadrilateral $ABCD$, $DC \parallel AB$, $AD = BC$, and AD is not parallel to BC. Through B, draw a line parallel to AD meeting ray DC at E. Since $ABDE$ has two pairs of opposite parallel sides, it is a parallelogram. Because the opposite angles of a parallelogram are equal, $\angle A = \angle BEC$. The opposite sides of a parallelogram are also equal, so $AD = BC = BE$. If two sides of a triangle are equal, the triangle is isosceles, which implies that $\angle BEC = \angle BCE$. Finally, because $DC \parallel AB$, $\angle BCE = \angle ABC$. The chain of equal angles now reads $\angle A = \angle BEC = \angle BCE = \angle ABC$. This means that $ABCD$ has a pair of consecutive equal angles on one of its parallel sides. By Theorem 5a, $ABCD$ is an isosceles trapezoid.



d. If a line perpendicularly bisects two sides of a quadrilateral, the quadrilateral is an isosceles trapezoid.
Proof: The two sides can't be consecutive, because if they were, you would have two consecutive parallel sides, which is impossible. In quadrilateral $ABCD$, l is the perpendicular bisector of AB and DC. Under reflection in l, therefore, $D' = C$ and $A' = B$. Thus, l is a line of symmetry and $ABCD$ is an isosceles trapezoid.



e. If two sides of a quadrilateral are parallel, and if the diagonals are equal, the quadrilateral is an isosceles trapezoid.
Proof:



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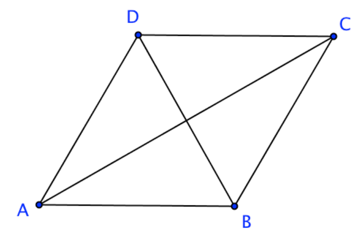
In quadrilateral $ABCD$, $DC \mid\mid AB$ and $AC = BD$. Let M be the midpoint of BC. Rotate $ABCD$ 180˚ around M. Since $B' = C$, $C' = B$, $BD' \mid\mid DC$, and $CA' \mid\mid AB$, A' is on ray DC and D' is on ray AB. Because rotation preserves segment length, $BD = CD'$. Therefore, $AC = BD = CD'$. Consider $\Delta ACD'$. Since two sides are equal, it is isosceles, so $\angle BAC = \angle BD'C$. BD is also rotated 180˚ around M, so $D'C \mid\mid BD$. Using transversal AD', we see that $\angle BD'C = \angle ABD$. Thus $\angle BAC = \angle BD'C = \angle ABD$. Because two angles in ΔABE are equal, the triangle is isosceles, which implies that $AE = BE$. In other words, E is equidistant from A and B, so it must lie on the perpendicular bisector of AB. A similar argument shows that E lies on the perpendicular bisector of DA. Since the perpendicular bisectors of AB and DC pass through the same point E, they coincide. By Theorem 5d, $ABCD$ is an isosceles trapezoid.

6. Rhombus: A quadrilateral with two lines of symmetry passing through opposite vertices. (So, a rhombus is a kite in two different ways.)
   a. If the diagonals of a quadrilateral perpendicularly bisect each other, the quadrilateral is a rhombus.
   Proof: By the definition of reflection, the two vertices not on either diagonal are images of each other under reflection in that diagonal. Therefore, both diagonals are lines of symmetry, which is the definition of a rhombus.

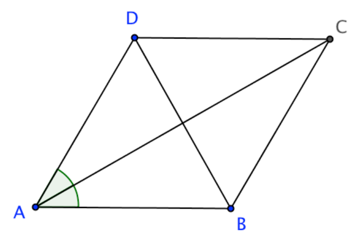
   b. If a quadrilateral is equilateral, it is a rhombus.
   Proof: Since opposite sides are equal, the quadrilateral is a parallelogram. Therefore, the diagonals bisect each other. Since two disjoint pairs of consecutive sides are equal, the quadrilateral is a kite. Therefore, the diagonals are perpendicular. Now we know that the diagonals perpendicularly bisect each other, so the quadrilateral is a rhombus by Theorem 6a.

c. If both diagonals of a quadrilateral bisect a pair of opposite angles, the quadrilateral is a rhombus.
Proof: Consider the diagonals separately. Since $\angle ABC$ and $\angle ADC$ are bisected, $ABCD$ is a kite with $AD = CD$ and $AB = CB$. Since $\angle BAD$ and $\angle BCD$ are bisected, $ABCD$ is a kite with $AD = AB$ and $CD = CB$. Therefore, all four sides are equal and the quadrilateral is a rhombus by Theorem 6b.

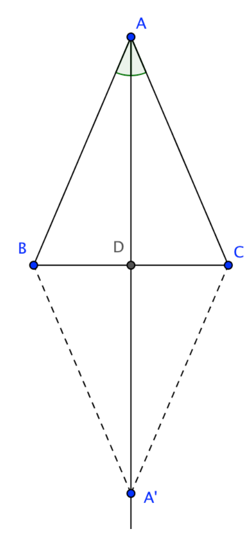


d. If both pairs of opposite sides of a quadrilateral are parallel, and if two consecutive sides are equal, the quadrilateral is a rhombus.
Proof: Since both pairs of opposite sides are parallel, the quadrilateral is a parallelogram. Therefore, both pairs of opposite sides are equal. Since a pair of consecutive sides are equal, all four sides must be equal. Hence the quadrilateral is a rhombus by Theorem 6b.

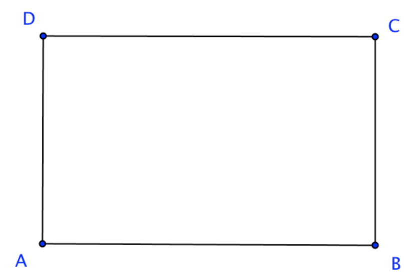
e. If a diagonal of a parallelogram bisects an angle, the parallelogram is a rhombus.
Proof: In parallelogram $ABCD$, diagonal AC bisects $\angle DAB$. The opposite sides of a parallelogram are parallel, so this implies that $\angle DCB$ is bisected as well by angle properties of parallel lines. By Theorem 4b, $ABCD$ is a kite. Therefore, $ABCD$ is a rhombus by Theorem 6d.



Now we are ready to prove Theorem 1e: If an angle bisector of a triangle is also a median, the triangle is isosceles.
Proof: In ΔABC, ray AD bisects $\angle BAC$ and $BD = CD$. Rotate ΔABC 180˚ around D. Because D is the midpoint of BC, $B' = D$ and $D' = B$. Because rotation preserves distance, $AD = A'D$. Now the diagonals of quadrilateral $ABA'C$ bisect each other, so $ABA'C$ is a parallelogram. But diagonal AA' bisects $\angle BAC$, so by Theorem 6e, $ABA'C$ is a rhombus. A rhombus is equilateral, so $AB = AC$.

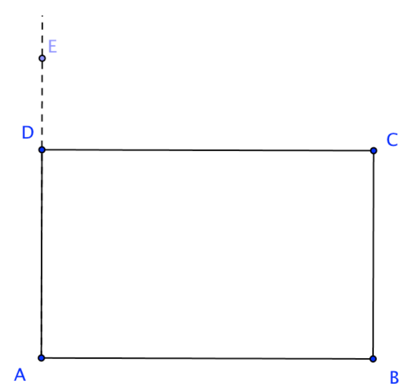


7. Rectangle: A quadrilateral with two lines of symmetry passing through midpoints of the opposite sides. (So, a rectangle is an isosceles trapezoid in two different ways.)
   a. If a quadrilateral is equiangular, it is a rectangle.
      Proof: Because $\angle A = \angle B$ and $\angle C = \angle D$, $ABCD$ is an isosceles trapezoid with line of symmetry through midpoints of AB and DC by Theorem 5b. Similarly, $\angle A = \angle D$ and $\angle B = \angle C$, so $ABCD$ is an isosceles trapezoid with line of symmetry through midpoints of AD and BC. By definition, $ABCD$ is a rectangle.

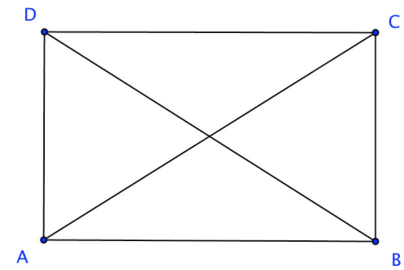


   b. If a parallelogram has a right angle, then the parallelogram is a rectangle.
      Proof: Suppose $\angle A = 90^\circ$. Then $\angle C = 90^\circ$ because opposite angles of a parallelogram are equal. The sum of the interior angles of a quadrilateral is 360°, which leaves a total of 180° for $\angle B$ and $\angle D$. Since they are also equal, they must be right angles as well. Hence all angles are equal right angles and the quadrilateral is a rectangle by Theorem 7a.

   c. An isosceles trapezoid with a right angle is a rectangle.
      Proof: Suppose $ABCD$ is an isosceles trapezoid with line of symmetry passing through bases AB and DC. Without loss of generality, we can suppose that $\angle A = 90^\circ$. Because the bases of an isosceles trapezoid are parallel $\angle EDC = \angle A = 90^\circ$. $\angle ADC$ and $\angle EDC$ are supplementary, so $\angle ADC = 90^\circ$ also. We also know that two consecutive angles of an isosceles trapezoid on the same base are equal, so $\angle B = \angle A = 90^\circ$and $\angle C = \angle ADC = 90^\circ$. Now $ABCD$ is equiangular, so by Theorem 7a it is a rectangle.



d. If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.
Proof: Because *ABCD* is a parallelogram, *AB* || *DC* and diagonals *AC* and *BD* bisect each other. Since the diagonals are equal as well, *ABCD* is an isosceles trapezoid whose line of symmetry passes through midpoints of *AB* and *DC* by Theorem 5e. By the same argument with parallel sides *AD* and *BC*, *ABCD* is an isosceles trapezoid whose line of symmetry passes through midpoints of *AD* and *BC*. It follows that *ABCD* satisfies the symmetry definition of a rectangle.



8. Square: A quadrilateral with four lines of symmetry: two passing through opposite vertices and two passing through interior points of opposite sides.
   a. A rectangle with consecutive equal sides is a square.
   Proof: A rectangle has two lines of symmetry passing through the opposite sides. The opposite sides of a rectangle are equal, so if two consecutive sides are also equal, it is equilateral. An equilateral quadrilateral is a rhombus, so its diagonals are additional lines of symmetry. Therefore, the rectangle is a square.

   b. A rhombus with consecutive equal angles is a square.
   Proof: The diagonals of a rhombus are lines of symmetry. The opposite angles of a rhombus are equal, so if two consecutive angles are also equal, it is equiangular. An equiangular quadrilateral is a rectangle by Theorem 7a, so it has two additional lines of symmetry passing through opposite sides. Therefore, the rhombus is a square

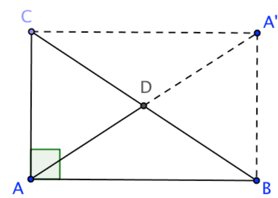
   c. An equilateral quadrilateral with a right angle is a square.
   Proof: An equilateral quadrilateral is a rhombus by Theorem 6b. Opposite angles of a quadrilateral are equal and the sum of the angles is 360˚, so all angles are right angles and the quadrilateral is also equiangular. An equiangular quadrilateral is a rectangle by Theorem 7a. If a quadrilateral is both a rhombus and a rectangle, it has four lines of symmetry and is therefore a square.

   d. An equiangular quadrilateral with consecutive equal sides is a square.
   Proof: An equiangular quadrilateral is a rectangle by Theorem 7a. The opposite sides of a rectangle are equal, and if consecutive sides are also equal, it must be equilateral. An equilateral quadrilateral is a rhombus by Theorem 6b. If a quadrilateral is both a rhombus and a rectangle, it has four lines of symmetry and is therefore a square.

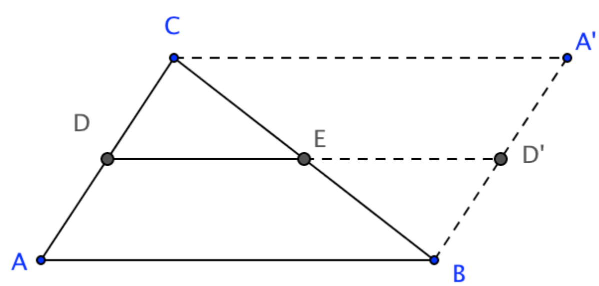
e. A quadrilateral with 4-fold rotational symmetry is a square.
Proof: Since a quadrilateral has four sides, consecutive sides and angles must map to each other under a 90˚ rotation. Because rotations preserve sides and angles, the quadrilateral must be both equilateral and equiangular, which implies that it is both a rectangle and a rhombus. If a quadrilateral is both a rhombus and a rectangle, it has four lines of symmetry and is therefore a square.

9. Additional Triangle Theorems

a. The median to the hypotenuse of a right triangle has half the length of the hypotenuse.
Proof: In right triangle ABC, $BD = CD$. Rotate ΔABC and median AD 180˚ around D. Because rotations preserve segment length and the rotation is 180˚, D is the midpoint of AA' as well as BC. Because the diagonals of quadrilateral $ABA'C$ bisect each other, it is a parallelogram by Theorem 3a. But a parallelogram with a right angle is a rectangle by Theorem 7b, and the diagonals of a rectangle are equal. Thus $AD = \frac{1}{2}AA' = \frac{1}{2}BC$.

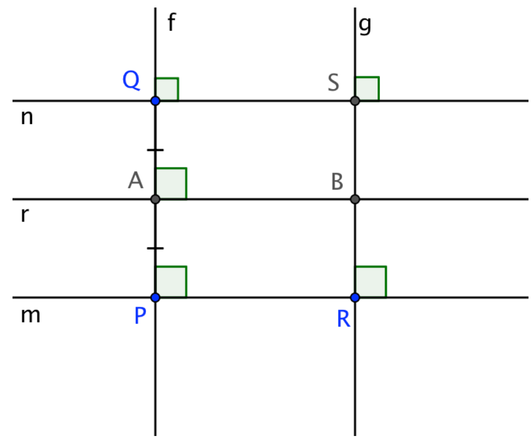


b. A segment joining the midpoints of two sides of a triangle (called a midsegment) is parallel to the third side and half as long.
Proof: In triangle ABC, D and E are midpoints of AC and BC respectively. Rotate ΔABC and segment DE 180˚ around point E. Since E is a midpoint, $B' = C$ and $C' = B$. Therefore, quadrilateral $ABA'C$ has 2-fold rotational symmetry, so by definition, it is a parallelogram.



Because rotation preserves segment length and D is a midpoint, $AD = DC = D'B$. But AD is parallel to BD' as well, so $ABD'D$ is also a parallelogram by Theorem 3d. The opposite sides of a parallelogram are parallel, so $DE \parallel AB$. Because rotation preserves length, $DE = ED' = \frac{1}{2}DD'$, and because the opposite sides of a parallelogram are equal, $DD' = AB$. Hence $DE = \frac{1}{2}AB$.
Note: The proof is shorter and more elegant using dilation. (See Chapter 4)

10. Given two distinct parallel lines *m* and *n*, there is a reflection mapping one to the other.



Proof: Draw line *f* perpendicular to *m* at *P* intersecting *n* at *Q*. *f* is also perpendicular to *n* by Theorem 15 of Chapter 1. (A line perpendicular to one of two distinct parallel lines is perpendicular to the other.) Draw *r*, the perpendicular bisector of segment *PQ*. Under reflection in *r*, *P′* = *Q* by definition. We will show that any other point *R* on line *m* reflects in *r* to a point on line *n*. Through *R*, draw line *g* perpendicular to *m*; *g* will also be perpendicular to *n*. Since corresponding angles are equal, *g* is parallel to *f*. By Theorem 15 again, *g* is perpendicular to *r*. *ABRP* and *QSAB* are parallelograms with a right angle, so by Theorem 7b they are rectangles. Chapter 2, Theorem 8f says that the opposite sides of a rectangle are equal, so *RB* = *AP* = *QA* = *SB*. Since *r* is the perpendicular bisector of segment *RS*, *R* reflects into *S*, which is on line n.

<u>Corollary</u>: Parallel lines are everywhere equidistant.

<u>Proof</u>: *PQ* = *RS* since *PA* + *AQ* = *RB* + *BS*.

Chapter 4: Dilation and Similarity

To begin this chapter, we expand on the foundational ideas of *Chapter 1 – Basics of Isometries and Congruence*.

Definitions

Dilation: A dilation with center O and scale factor $k \neq 0$ maps O to itself and any other point P to P' so that O, P, and P' are collinear and the directed segment $OP' = k \cdot OP$. (If k is negative, rays OP' and OP point in opposite directions.)

Similarity: Two figures are *similar* if one can be superposed on the other by a dilation followed by a sequence of isometries.

General Trapezoid: A quadrilateral where one side is a dilation or translation of the opposite side. If a translation, the trapezoid is a parallelogram because one pair of opposite sides is both parallel and equal. (Under our inclusive definitions, a parallelogram is a special trapezoid.)

Note: The properties of a general trapezoid are proved in Theorem 15. These properties were not listed in Chapter 2 because the definition of a general trapezoid depends on dilation.

Postulates

To our previous list (the parallel postulate, reflection is an isometry, and the construction postulates), we add one necessary assumption about dilation:

6. Dilation preserves collinearity.

Basic Theorems

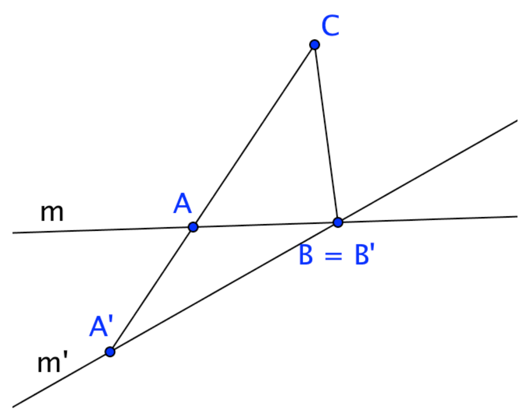
As in Chapter 1, we use an asterisk to indicate theorems that in our view should be discussed informally, rather than proved formally when working with students.

1. * A dilation with scale factor -1 is a half-turn around the center of dilation.
   Proof: This follows immediately from the definitions of dilation and rotation.
2. * A dilation with scale factor $k < 0$ is the composition of a dilation with scale factor -1 and a dilation with scale factor $|k|$, all with the same center.
   Proof: This is another immediate consequence of the definitions.
3. * If O is the center of a dilation, k is the scale factor, and O, A, and B are distinct collinear points, then $A'B' = |k|AB$.
   Proof:
   Case 1: A and B are on the same side of O.
   By the definition of dilation and the given information, O, A, A', B, and B' are all collinear. Even if k is negative, A', and B' are on the same side of O. Suppose, without loss of generality, that A is between O and B, so that $OA + AB = OB$. Hence $OA < OB$ and, even if k is negative, $OA' = |k|OA < |k|OB = OB'$. Therefore, A' is between O and B' and $OA' + A'B' = OB'$. By the definition of dilation, $OB' = |k|OB$ and $OA' = |k|OA$. Therefore, $A'B' = OB' - OA' = |k|OB - |k|OA = |k|(OB - OA) = |k|AB$.
   Case 2: A and B are on opposite sides of O.
   The argument is very similar, except that one starts with $AO + OB = AB$. It's still the

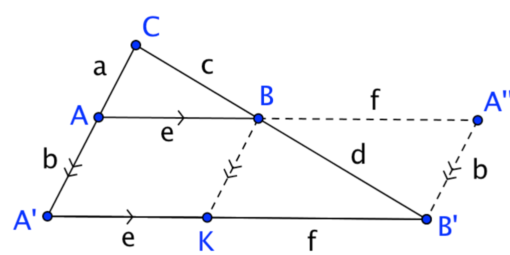
case that O, A, A', B, and B' are all collinear, but now O is between A' and B'.
Case 3: Either A or B is the same as O.
In this case, $A'B' = |k|AB$ by the definition of dilation.

4. * The image of a line under a dilation is a line.
Proof: Let O be the center of a dilation. If the pre-image line m contains O, then the image will be the same line with points mapped $|k|$ times their distance from O, on the same side if $k > 0$ and on the other side if $k < 0$. So, $m' = m$ in this case. Now suppose m does not contain O. Since dilations preserve collinearity, m' will be a possibly proper subset of a line. Neither m nor the line containing m' can be parallel to OP. Pick any point Q on the line containing m' and draw line OQ. Because neither m nor m' is parallel to line OQ, it will intersect m at P. Since the dilation maps P to Q, any point on the line containing m' is an image point. Therefore, the image of m is an entire line.

5. Fundamental Theorem of Dilations (FTD): If C, A, and B are not collinear, the segment $A'B'$ joining the images of A and B under a dilation with center C and scale factor k is parallel to segment AB and has length $|k|AB$.
Proof:
Part 1: Segment $A'B'$ || segment AB.
Proof: Let the image of line m through points A and B be m'. Assume that m and m' are not parallel. Then they meet at a point $B \neq C$. Choose another point A on m. The image of A is the intersection of line CA with line m'. B is its own image, so $B' = B$. From A and A', $k \neq 1$. From B and B', $k = 1$. This contradiction completes the proof.



Part 2: If $k > 0$, $A'B' = kAB$ and the directed segments $A'B'$ and AB point in the same direction.
Proof: Let A' and B' be the images of A and B respectively, with the lengths of segments CA, CA', CB, and CB' as labeled in the figure. From the definition of dilation, $k = \frac{a+b}{a} = \frac{c+d}{c}$. By Part 1, AB and $A'B'$ are parallel. Draw a line through B parallel to AA' intersecting $A'B'$ at K. Now dilate ΔABC from center B with scale factor $-\frac{d}{c}$, resulting in the figure shown. We indicate segment lengths obtained from knowing that opposite parallel sides of a quadrilateral insure that it's a parallelogram, and the opposite sides of a parallelogram are equal. By the definition of dilation, $\frac{d}{c} = \frac{f}{e}$. Therefore, $1 + \frac{d}{c} = 1 + \frac{f}{e}$, or $\frac{c+d}{c} = \frac{e+f}{e}$. In terms of the segments, this translates to $\frac{B'C}{BC} = \frac{A'B'}{AB} = k$, so $A'B' = kAB$, as desired.



Part 3: If $k < 0$, $A'B' = |k|AB$ and the directed segments $A'B'$ and AB point in opposite directions.
Proof: By Theorem 2, we can decompose the dilation to a composition of a half-turn around the center followed by a dilation from the center with scale factor $|k|$. The image of the directed segment AB under a half turn has the same length and points in the opposite direction. Now we apply Part 2 above to obtain the result.
Pedagogical note: The proof is challenging. If it you deem it to be too much for your students, you can consider this theorem to be an axiom, as it is in the Common Core State Standards. We present it as a theorem because, although it does not follow from the definition of dilation, it can be derived from the assumption that dilations preserve collinearity.

6. Under a dilation, $A'B' = |k|AB$. The directed segments AB and $A'B'$ point in the same direction if $k > 0$ and in opposite directions if $k < 0$.
   Proof: Theorem 3 proves this if O, A, and B are collinear. Theorem 5 (the FTD) proves it if not.
7. * Dilation preserves betweenness.
   Proof: If points A, B and C are in order on a line, $AB + BC = AC$. If k is the scale factor of the dilation, then $A'B' = |k|AB$, $B'C' = |k|BC$, and $A'C' = |k|AC$ by Theorem 5. Therefore, $A'B' + B'C' = A'C'$. This implies that B' is between A' and C'.
8. * The image of a segment under a dilation is a segment and the image of a ray is a ray.
   Proof: Let A and B be two points with images A' and B' under a dilation with center O and scale factor k. Because collinearity and betweenness are preserved, the image of segment AB is a subset segment $A'B'$, and similarly for rays. Any point Q on segment $A'B'$ or ray A'B' is the image of a point Q' under a dilation with center O and scale factor $\frac{1}{k}$. Because collinearity and betweenness are preserved, Q' must lie on segment AB or ray AB. So, the image is the entire segment or ray.
9. Dilation preserves angle measure.
   Proof: The image of any ray under a dilation is another ray that is either parallel to, or collinear with, its pre-image. (If $k < 0$, the directions of both rays will be reversed.) The same is true for a translation and a half-turn (needed if $k < 0$). So, with a translation and a half-turn if $k < 0$, we can map any angle into its image under a dilation. Since translations and half-turns preserve angles, dilations do too.
10. Dilation preserves the ratio of the lengths of any two segments.
    Proof: Let AB and CD be two segments. $A'B' = |k|AB$ and $C'D' = |k|CD$ by Theorem 6. Therefore, $\frac{A'B'}{C'D'} = \frac{AB}{CD}$.

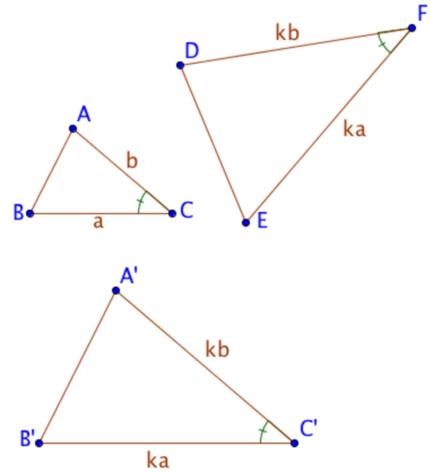
Similar Triangles

11. Similar triangles have congruent angles and proportional sides.
    Proof: Dilation preserves angle measure by Theorem 9 and isometries do also. Dilation preserves the ratio of side lengths by Theorem 10 and isometries do also because they preserve distance.

12. Similarity Criteria for Triangles

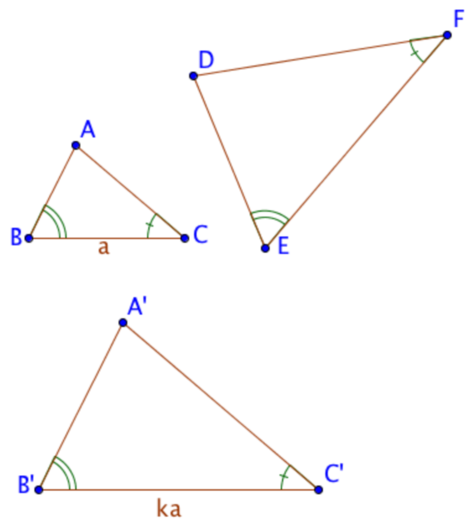
a. SSS Similarity: If the sides of two triangles are proportional, then the triangles are similar.

<u>Proof</u>: Assume ΔABC and ΔDEF have proportional sides, with ratio k as shown in the diagram. Dilate ΔABC with any center and scale factor k, yielding $\Delta A'B'C'$. $\Delta A'B'C'$ is congruent to ΔDEF by SSS congruence. Since we can map ΔABC to ΔDEF by a dilation and a sequence of isometries (those that map $\Delta A'B'C'$ to ΔDEF,) the triangles are similar by definition.



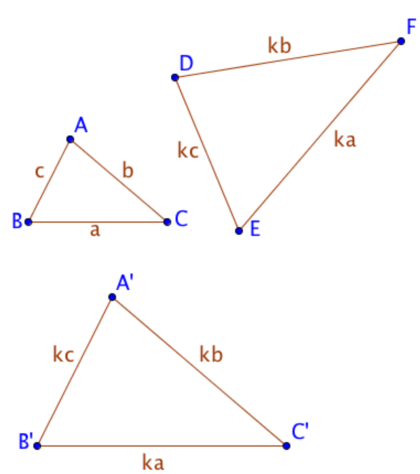
b. SAS Similarity: If a pair of sides in one triangle is proportional to a pair of sides in another triangle, and if the angles between those sides are congruent, then the triangles are similar.

<u>Proof</u>: Given ΔABC and ΔDEF such that $\frac{DF}{AC} = \frac{EF}{BC} = k$ and $\angle C = \angle F$. Dilate ΔABC with any center and scale factor k, yielding $\Delta A'B'C'$. ΔA'B'C' is congruent to ΔDEF by SAS congruence. Therefore, ΔABC and ΔDEF are similar by the same argument as above.



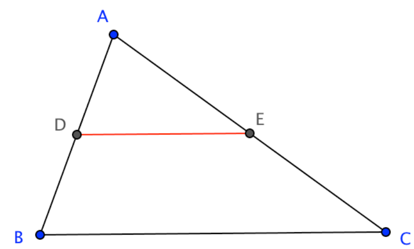
c. AA Similarity: If two angles in one triangle are equal to two angles in another triangle, the triangles are similar.

<u>Proof</u>: Given ΔABC and ΔDEF such that $\angle C = \angle F$ and $\angle B = \angle E$. Let $\frac{EF}{BC} = k$, so that $EF = ka$. Dilate ΔABC from any point with scale factor k. The resulting $\Delta A'B'C'$ is congruent to ΔDEF by ASA congruence. Therefore, ΔABC and ΔDEF are similar.

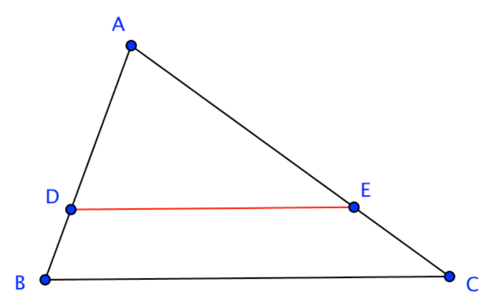


d. HL Similarity Criterion for Right Triangles: If the hypotenuse and one leg of one right triangle are proportional to the hypotenuse and one leg of another, then the right triangles are similar.
Proof: Given right ΔABC and right ΔDEF such that $\angle C = \angle F = 90^\circ$, and $\frac{DE}{AB} = \frac{EF}{BC} = k$. Dilate ΔABC from any point with scale factor k. The resulting $\Delta A'B'C'$ is congruent to ΔDEF by HL congruence for right triangles. Therefore, ΔABC and ΔDEF are similar.

13. A segment joining the midpoints of two sides of a triangle (called a *midsegment*) is parallel to the third side and half as long.
Note: This was proved in Chapter 3. We include this proof as well because it is shorter and more elegant. If you include Theorem 14 below in your development, you can just mention this theorem as a corollary of it.
Proof: Let DE be a midsegment of ΔABC. Dilate ΔABC from A with scale factor $\frac{1}{2}$. Since D and E are midpoints, $B' = D$ and $C' = E$. The FTD tells us that $DE || BC$ and $DE = \frac{1}{2}BC$.



14. If a segment joins points on two sides of a triangle whose distances are the same fraction k ($0 < k < 1$) of the distance from their common endpoint to their other endpoint, then the segment joining these points is parallel to the third side and its length is the same fraction k of it.
Proof: Let D and E be points on sides AB and AE of ΔABC respectively, chosen so that $AD = kAB$ and $AE = kAC$, where $0<k<1$. Dilate ΔABC from A with scale factor k. By the definition of dilation, B' = D and C' = E. The FTD tells us that $DE || BC$ and $DE = kBC$.
Note: A similar argument works if $k > 1$ if you replace segments AB and AC by rays AB and AC.



General Trapezoid

15. Properties of a General Trapezoid:
    a. A pair of opposite sides (called *bases* in the dilation case) are parallel.
    Proof: If one side is a dilation of another, the center of dilation can't be on a line containing one of these sides, because if it were, the opposite sides would be collinear. By the FTD, the image of a side is parallel to its pre-image. If one side is a translation of another, the image of a segment under translation by a vector not parallel to the line is a line parallel to its pre-image.
    b. Consecutive angles (on different bases if the trapezoid is not a parallelogram) are supplementary.
    Proof: This is a property of parallel lines cut by a transversal.

Chapter 5: Circles

Circle Definitions

Circle: The set of points that are a given fixed distance from a given point (the *center*).
Radius: Either the given fixed distance or a segment joining the center to a point on a circle.
Chord: A segment connecting two points on a circle.
Diameter: A chord that passes through the center of a circle. The word *diameter* also refers to the length of any such segment.
Secant: A line that intersects a circle in two points.
Tangent: A line that intersects a circle in exactly one point (the *point of tangency* or *point of contact*).
Tangent Segment: A segment that is a subset of a tangent line, with an endpoint on the circle.

Notation

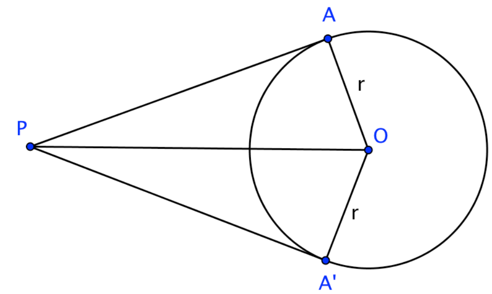
A circle can be named by its center if only one circle in the diagram has that center. In that case, *Circle O* names the circle whose center is O.

Theorems

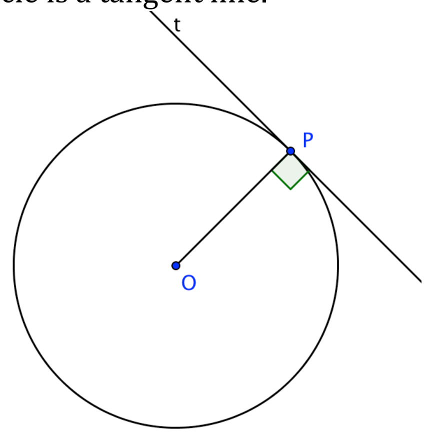
We include neither the inscribed angle theorem nor theorems about intersecting chords, because we did not find transformational proofs for them. They should certainly be included in a geometry course, but they can be proved using traditional methods.

1. Any diameter of a circle is a line of symmetry.
   Proof: Let P be a point on circle O and let d be a diameter. Consider reflection in d. The reflection of OP is OP', so $OP' = OP$ because reflection preserves distance. Therefore P' is also on circle O.
2. If a diameter is perpendicular to a chord, it bisects the chord.
   Proof: Let diameter d of circle O be perpendicular to chord AB. Since A and B are both on the circle, $OA = OB$. Therefore, ΔOAB is isosceles. By Theorem 1d of Chapter 2, its line of symmetry is a perpendicular bisector.
3. If a diameter bisects a chord, it is perpendicular to the chord.
   Proof: Let diameter d of circle O bisect chord AB at K. Since A and B are both on the circle, $OA = OB$. Therefore, ΔOAB is isosceles. By Theorem 1d of Chapter 2, its line of symmetry is a perpendicular bisector.
4. The perpendicular bisector of a chord passes through the center of the circle.
   Proof: The center of the circle is equidistant from the endpoints of the chord, so it lies on its perpendicular bisector.

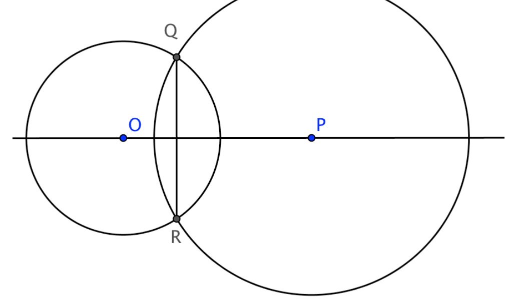
5. The reflection of a tangent segment in the segment joining its external endpoint to the center is another tangent segment.
 Proof: Let PA be a tangent segment to circle O as shown. Reflect PA in PO; its image is PA'. Since reflection preserves distance, $OA' = OA$, so A' is on circle O. No other point on the line containing PA is a distance r from O, so no other point on the line containing PA' can be r away from O. Therefore PA' is a tangent segment.



6. Tangent segments to a circle from an external point are equal.
 Proof: Using Theorem 5 and the same diagram, $PA' = PA$ because reflection preserves distance.
7. A segment joining an external point to the center of a circle bisects the angle formed by the two tangent segments drawn from that point.
 Proof: Using Theorem 5 and the same diagram, $\angle OPA' = \angle OPA$ because reflection preserves angle measure.
8. A line perpendicular to a radius at its endpoint on the circle is a tangent line.
 Proof: Let line t be perpendicular to radius OP at P. t reflects onto itself in OP. Suppose t intersected circle O at another point Q. Then Q' would be on both t and circle O as well. In that case, circle O and t would intersect in three points (Q, P and Q'), which contradicts Postulate 3. Therefore, t intersects circle O only at P, so it is a tangent line.



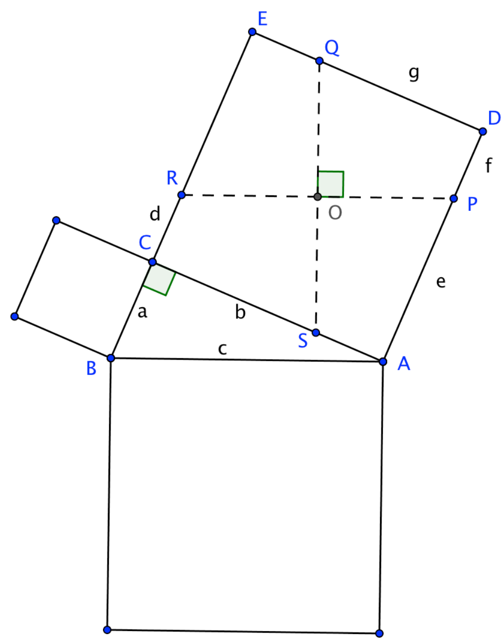
9. A tangent to a circle is perpendicular to a radius drawn to the point of tangency.
 Proof: Let u be tangent to circle O, with P as the point of tangency. Drop a perpendicular b from O to u. Let P' be the reflection of P in b. Since b is perpendicular to u, P' must be on u. But since reflections preserve distance, $OP = OP'$. Therefore P' is also on the circle. But u is tangent to the circle at one point, so $P' = P$. Since P is its own reflection in b, it must be on line b. Thus OP, the radius drawn to the point of tangency, is perpendicular to u.
10. When two circles intersect in two points, the line through their centers is the perpendicular bisector of their common chord.
 Proof: Circles O and P have common chord QR. Since O and P are both equidistant from Q and R, they lie on its perpendicular bisector.



Chapter 6: Pythagorean Theorem

There are many proofs of the Pythagorean theorem, including many dissection proofs. This one is tailor-made for transformations, because it relies on showing that quadrilaterals (not triangles) are congruent. Traditional curriculum doesn't include congruent quadrilaterals. A dissection proof is very visual: you literally cut the square built on the longer leg of a right triangle into congruent quadrilaterals and reassemble the pieces, along with the square built on the shorter leg, to build a square on the hypotenuse. We want to prove that this really works – that the five pieces really do form a square on the hypotenuse. Along the way, we use properties of a parallelogram and write simple equations relating some of the side lengths.

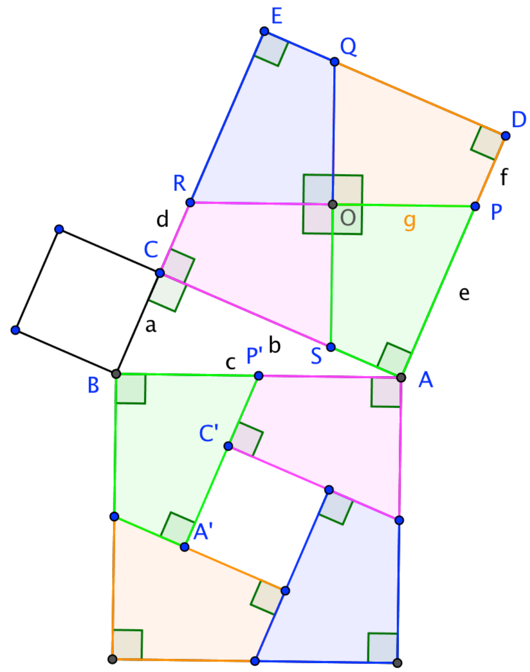
Henry Perigal, Jr**.** (1 April 1801 – 6 June 1898) was a British stockbroker and amateur mathematician. He provided this dissection in his booklet *Geometric Dissections and Transpositions* (London: Bell & Sons, 1891). He had the dissection printed on his business cards, and it also appears on his tombstone. (Source: Wikipedia)



The diagram shows a right triangle with squares built on the legs and hypotenuse. O is the center of the square ACED (the intersection of its diagonals). Draw lines through O parallel and perpendicular to BA and cut them off at points P, Q, R, and S on the square. There are four right angles with vertex O.

Square $ADEC$ has 4-fold rotational symmetry (Chapter 2 Theorem 9d). Therefore, under a 90˚ counterclockwise rotation around O, $A \to D \to E \to C \to A$. This implies that sides $AD \to DE \to EC \to CA \to AD$. In addition, line $PR \to$ line $QS \to$ line PR. Since P is the intersection of line PR and side AD, and since Q is the intersection of line QS and side DE, this means that $P \to Q$. A similar argument applies to the other points on the sides of the square, so $P \to Q \to R \to S \to P$. Since the rotation is around O, O remains fixed. Since vertices map to vertices and rotation maps segments to segments, we have shown that, under a 90˚ counterclockwise rotation around O, $OPDQ \to OQER \to ORCS \to OSAP \to OPDQ$. By definition, all four of these quadrilaterals are congruent. Note that each quadrilateral has two opposite right angles.

Translate each of these quadrilaterals to a corner of the square built on the hypotenuse, as shown in the diagram. By construction, $RP \parallel BA$. $AD \parallel BE$ because the opposite sides of a square are parallel. By Chapter 2 Theorems 9a and 7e and Chapter 3 Theorem 3b, *BAPR* is a parallelogram. Its opposite sides are equal by Chapter 2 Theorem 4f. Therefore, $a + d = e$.



But translation preserves segment length, so $A'P' = e$ and $C'P' = d$. Thus $A'C' = A'P' - C'P' = e - d = a$. The same argument applies to the other sides of the central white quadrilateral, so all sides have length a and all angles are right angles, so it is a square with area a^2. (It is possible, but not necessary for this proof, to show it is the translation image of the original square on side a.)

By the rotational symmetry, $OP = OQ = OR = OS = g$, and since the opposite sides of a parallelogram are equal, $2g = c$. Since each quadrilateral has two right angles and the sum of the angles in a quadrilateral is 360˚, the other two angles are supplementary. Since the four quadrilaterals are congruent, $\angle BP'C'$ and $\angle AP'C'$ are supplementary, so BP' and $P'A$ are collinear. The same argument applies to the other midpoints of sides of the square on hypotenuse *AB*. Therefore, the five pieces cover the square on the hypotenuse with no gaps or overlaps. It follows that the diagram shows that the area of the square built on the hypotenuse is the sum of the areas of the squares built on the legs.

Further Exploration:
It is possible to find the lengths *e* and *f* in terms of *a* and *b*. This is an interesting exercise.

Appendix

An asterisk indicates a theorem which, in our view, should be discussed, but need not be proved formally in a standard geometry course. (See Chapter 1 for an explanation.)

List of Theorems from Chapter 1

Basic Theorems

1. * If $A' = B$ under a reflection, then $B' = A$.
2. * Reflection preserves collinearity and betweenness.
3. * If $A \rightarrow A'$ and $B \rightarrow B'$ under a reflection, segment AB must map onto segment $A'B'$.
4. * Reflections map rays onto rays and lines onto lines.
5. * Congruent segments have equal length. Congruent angles have equal measure.
6. * The corresponding sides and angles of congruent polygons have equal measure.
7. * There is a reflection that maps any given point P onto any given point Q.

Triangle Congruence

8. A point P is equidistant from two points A and B if and only if it lies on their perpendicular bisector.
9. If two segments AB and CD have equal length, then one is the image of the other, with C the image of A and D the image of B, under either one or two reflections.
10. Equal length segments are congruent. If we combine this with Theorem 5, we have: Segments are congruent if and only if they have equal length.
11. Congruence Criteria for Triangles
    a. SSS Congruence: If all sides of one triangle are congruent, respectively, to all sides of another, then the triangles are congruent.
    b. SAS Congruence: If two sides of one triangle are congruent to two sides of another, and if the included angles have equal measure, then the triangles are congruent.
    c. ASA Congruence: If two angles of one triangle are congruent to two angles of another, and if the sides common to these angles in each triangle are congruent, then the triangles are congruent.
12. HL Congruence Criterion for Right Triangles: If the hypotenuse and one leg of one right triangle are congruent to the hypotenuse and one leg of another, then the right triangles are congruent.
13. If two triangles are congruent, one can be superimposed on the other by a sequence of at most three reflections.
14. * Angles with equal measure are congruent. If we combine this with Theorem 5, we have: Angles are congruent if and only if they have equal measure.

Two Reflections

15. * If a line is perpendicular to one of two parallel lines, it is perpendicular to the other.
16. The composition of two reflections in parallel lines is translation. The translation vector is perpendicular to the lines, points from the first line to the second, and has length twice the distance between the lines. This implies that any translation can be decomposed into two reflections.

17. The composition of two reflections in intersecting lines is a rotation around their point of intersection. The angle of rotation is twice the directed angle between the lines going from the first reflection line to the second (either clockwise or counterclockwise). This implies that any rotation can be decomposed into two reflections.
18. * Reflection, rotation, and translation preserve collinearity, betweenness, segment length and angle measure.
19. * If $A \rightarrow A'$ and $B \rightarrow B'$ under a reflection, rotation, or translation, segment AB must map onto segment $A'B'$.
20. * Reflections, rotations, and translations map rays onto rays and lines onto lines.
21. * Given two intersecting lines, there are two reflections mapping one to the other. Note: The related theorem for distinct parallel lines is Theorem 10 of Chapter 3.

Translations, Half-Turns, and Parallels

22. * If $A' = B$ under a half-turn, then $B' = A$.
23. * The image of segment AB under a half-turn around its midpoint is the segment BA. That is, $A' = B$ and $B' = A$.
24. * A line is its own image under a half-turn around a point on the line.
25. The image of a line under a half-turn is parallel to the pre-image.
26. * When two lines intersect, the vertical angles are equal.
27. * If two distinct lines are cut by a transversal, they are parallel if and only if the alternate interior angles are equal.
28. * If two distinct lines are cut by a transversal, they are parallel if and only if the corresponding angles are equal.
29. * The composition of translations is commutative.
30. * The translation image of a line is parallel to the line.
31. * Any representative of a vector can be superimposed on any other by a translation.
32. * If two distinct lines are cut by a transversal, an angle on one line is the translation image of an angle on the other if and only if the lines are parallel.

Sum of Angles

33. The sum of the angles of a triangle is 180˚. An exterior angle of a triangle is equal to the sum of the remote interior angles. Therefore, it is greater than either one.
34. The sum of the interior angles of a quadrilateral is 360˚. (A concave quadrilateral will have an interior angle greater than 180˚.)

List of Theorems from Chapter 4

Basic Theorems

1. * A dilation with scale factor -1 is a half-turn around the center of dilation.
2. * A dilation with scale factor $k < 0$ is the composition of a dilation with scale factor -1 and a dilation with scale factor $|k|$, all with the same center.
3. * If O is the center of a dilation, k is the scale factor, and O, A, and B are distinct collinear points, then $A'B' = |k|AB$.
4. * The image of a line under a dilation is a line.
5. <u>Fundamental Theorem of Dilations (FTD)</u>: If C, A, and B are not collinear, the segment $A'B'$ joining the images of A and B under a dilation with center C and scale factor k is parallel to segment AB and has length $|k|AB$.
6. Under a dilation, $A'B' = |k|AB$. The directed segments AB and $A'B'$ point in the same direction if $k > 0$ and in opposite directions if $k < 0$.
7. * Dilation preserves betweenness.
8. * The image of a segment under a dilation is a segment, the image of a ray is a ray.
9. Dilation preserves angle measure.
10. Dilation preserves the ratio of the lengths of any two segments.

Similar Triangles

11. Similar triangles have congruent angles and proportional sides.
12. Similarity Criteria for Triangles
    a. SSS Similarity: If the sides of two triangles are proportional, then the triangles are similar.
    b. SAS Similarity: If a pair of sides in one triangle is proportional to a pair of sides in another triangle, and if the angles between those sides are congruent, then the triangles are similar.
    c. AA Similarity: If two angles in one triangle are equal to two angles in another triangle, the triangles are similar.
    <u>Note</u>: These correspond to the congruence criteria. AA Similarity corresponds to ASA Congruence. The side isn't needed because it takes two or more sides to make a proportion.
13. A segment joining the midpoints of two sides of a triangle (called *a midsegment*) is parallel to the third side and half as long.
14. If a segment joins points on two sides of a triangle whose distances are the same fraction k ($0 < k < 1$) of the distance from their common endpoint to their other endpoint, then the segment joining these points is parallel to the third side and its length is the same fraction k of it.

General Trapezoid

15. Properties of a General Trapezoid:
    a. A pair of opposite sides (called *bases* in the dilation case) are parallel.
    b. Consecutive angles (on different bases if this is not a parallelogram) are supplementary.

List of Theorems from Chapter 5

1. Any diameter of a circle is a line of symmetry.
2. If a diameter is perpendicular to a chord, it bisects the chord.
3. If a diameter bisects a chord, it is perpendicular to the chord.
4. The perpendicular bisector of a chord passes through the center of the circle.
5. The reflection of a tangent segment in the segment joining its external endpoint to the center is another tangent segment.
6. Tangent segments terminating on a circle from an external point are equal.
7. A segment joining an external point to the center of a circle bisects the angle formed by the two tangent segments drawn from that point.
8. A line perpendicular to a radius at its endpoint is a tangent line.
9. A line not perpendicular to a radius at its endpoint is not a tangent line. (Theorems 8 and 9 can be combined: A line is perpendicular to a radius at its endpoint if and only if it is a tangent line.)
10. When two circles intersect in two points, the line through their centers is the perpendicular bisector of their common chord.