Computing Transformations

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Teacher Notes

These lessons provide the framework of an approach to computing geometric transformations in a precalculus level course, starting with complex numbers, and ending with matrices. The end result is a deeper understanding of complex numbers, and some facility and intuition about the basics of matrix arithmetic. The geometric transformations serve as the motivation for this work, which is otherwise often devoid of interest for many students.

The lessons assume a fair amount of teacher explanation and class discussion, and should not be expected to teach themselves. The unit is based on material I taught at The Urban School of San Francisco, in my Space class, an advanced high school geometry elective. The course and this material evolved over a dozen iterations. See <www.picciotto.org/math-ed/space>.

Prerequisites

Students should have been introduced to complex numbers, and be familiar with their rectangular and polar representations. At The Urban School, that introduction happens towards the end of our Algebra 2 course. See our Complex Numbers unit at <www.picciotto.org/math-ed/alg-2/complex.pdf>.

Students should also have been introduced to matrices and matrix multiplication. I used the approach in the University of Chicago School Mathematics Project *Advanced Algebra* book.

Technology

Some lessons assume the use of GeoGebra. It is of course possible to do much of this work with other software or devices. In fact, I used the TI-89 calculator when teaching this material. This document is my attempt to adapt the original lessons to GeoGebra, but it has not been tried with actual students, so proceed with caution. If you create a version of this packet using some other technological support, I will gladly credit you and post it alongside this version and the TI-89 version on <www.MathEducationPage.org>.

Review: Slope of Perpendicular Lines

This lesson reviews the "opposite reciprocal" result, but also proves it. It also includes a reminder of similar triangles. All of this will be needed in the subsequent pages.

Complex Numbers Basics (and Transformations)

This lesson combines a (perhaps too sketchy) review of complex numbers with an introduction to two key questions we address in this unit:
◊ How do we compute the coordinates of images of points under various transformations?
◊ How do we use composition of transformations we know how to compute, in order to figure out more challenging computations?

Note that #3 does not require knowing about multiplication in polar form. (This fact will be important in the next two lessons, because we will use it to prove a key result about that.)

#5. Multiply by $(1, \theta)$ -- a complex number on the unit circle

#6. Subtract $p+qi$, multiply by $(1, \theta)$, add $p+qi$. In other words, translate to the origin, rotate around the origin, translate back.

Complex Multiplication in Polar Form

This two-part lesson ends with a proof of the geometric interpretation of complex number multiplication. One way to teach it is to have the students work through the first page on graph paper, providing support as needed. Then, present the key ideas of the second page, culminating with the proof, without the worksheet. And finally have the students work through the second page themselves. This figure may be useful on a projector or interactive white board:
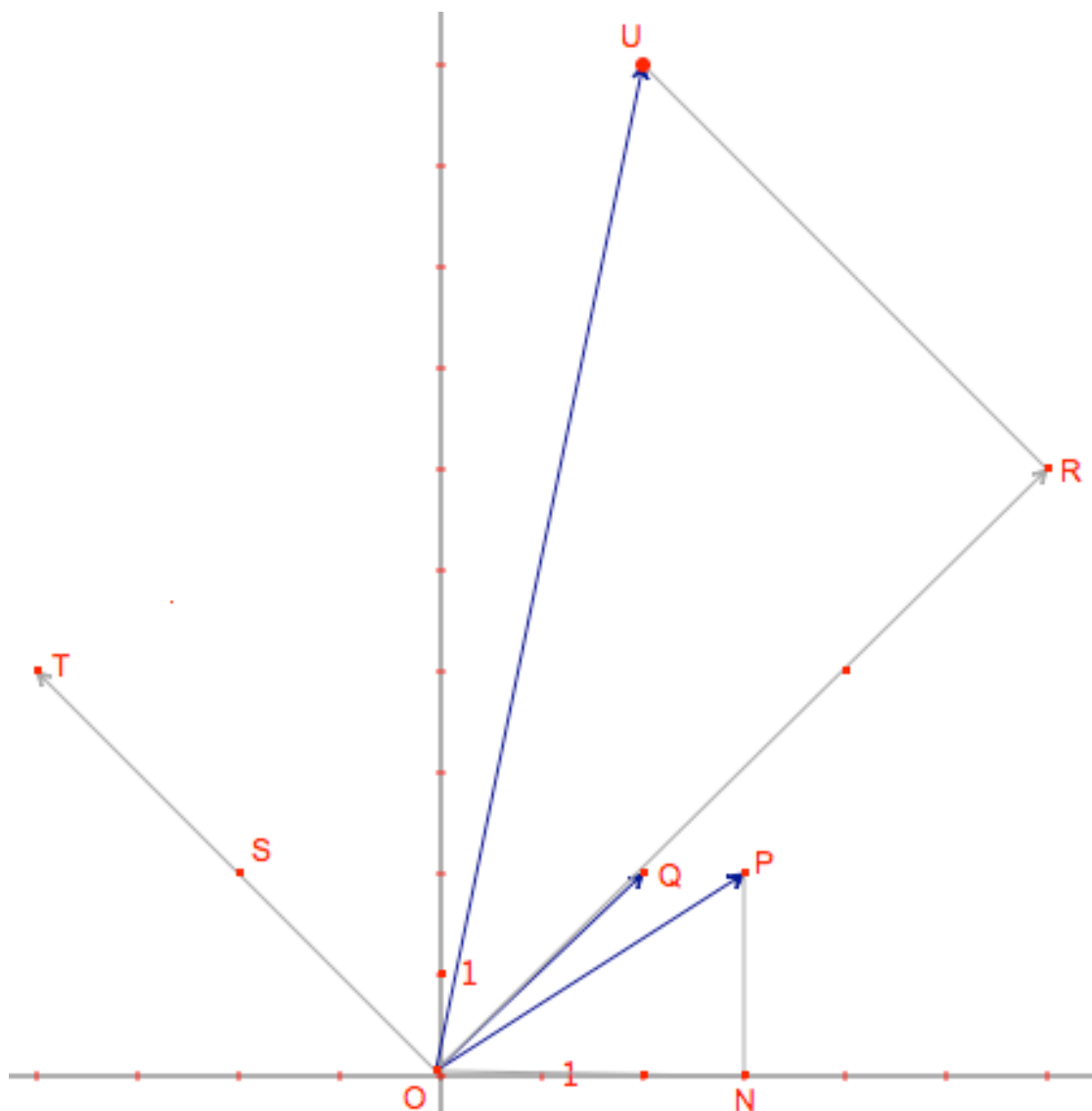
See also the corresponding Cabri figure at <www.picciotto.org/math-ed/space>.

The heart of the proof is SAS similarity between ΔONP and ΔORU. Avoid a circular argument! U = P·R follows from the argument in #2 of the first page which is generalized in #1 on the second page. The right angle at ∠R follows from the argument involving slopes that we used in the "Complex Numbers" lesson.

Computing Any Isometry Using Complex Numbers

The five-step method for #7 is: translate so the y-intercept is at the origin, rotate so the line coincides with the x-axis, reflect, rotate back, translate back. Assuming $\theta = \tan^{-1}(2)$, this calculation will do it:

$$(conjugate(((3+2i) - 5i) (cos(-\theta) + i \sin(-\theta))))(cos(\theta) + i \sin(\theta)) + 5i$$

Matrices

The remaining lessons on the one hand replicate the approach we used with complex numbers, using matrices, and on the other hand introduce increasingly efficient ways to use GeoGebra to actually do the computations.

Having students themselves find the matrices for the transformations is essential if you want students to understand why this all works. The fact that this works so well is very motivating as an introduction to matrices. This, and demystifying computer graphics (including interactive geometry,) is what this unit is about. It is not so much about mastering a supposedly useful tool to carry out the transformations, as that is already automated in the software we are using.

Review: Slope of Perpendicular Lines

0. What are the three ways to show triangles are similar?

1. Consider two perpendicular lines L and M. What is the relationship between their slopes?
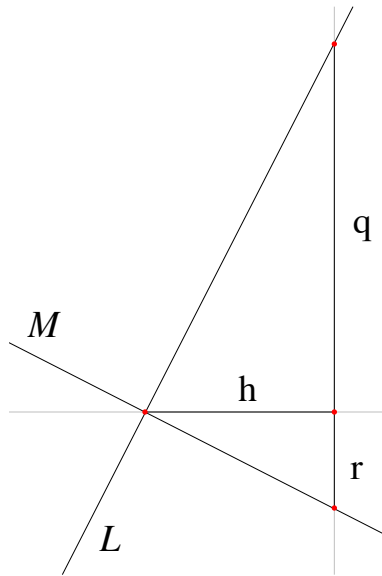
2. Use the slope triangles in the figure above to prove this. Hint: start by proving the triangles are similar. (Slope triangles by definition have perpendicular sides. The labels h, q, and r represent the lengths of the segments.)

3. Now prove the converse of this result. ("If the slopes of two lines are … then the lines are perpendicular.") Hint: you can do it by drawing two congruent right triangles.

Complex Numbers Basics (and Transformations)

You can think of a complex number as a vector that starts at the origin.

Rectangular form: a+bi, with $i^2 = -1$.
a is the real part. b is the imaginary part.

Polar form: (r, θ)
r is known as the radius, the modulus, the magnitude, or the absolute value of the number.
θ is the angle or argument.

1. Use basic trig to convert from polar to rectangular form and vice versa. (Draw an example on graph paper to help you remember how to do this. Work out examples in each of the four quadrants.)

GeoGebra shortcuts:
  ◊   Converting from rectangular to polar: for example ToPolar[3+4i]
  ◊   Converting from polar to rectangular: it's simply r·cos(θ)+r·sin(θ)·i

2. Complex addition: (a+bi)+(c+di).
    a. Rearrange the terms so that the real part is first, and the imaginary part last.
    b. Draw an example of this on graph paper, and explain how this addition works like vector addition.
    c. How would you use complex numbers to find the image of (a, b) by a *translation* (v, w)?

3. Complex multiplication: simple cases. For each example below, draw an example on graph paper.
    a. If k is a real number, distribute k(c+di). How does the resulting vector (the result of your calculation) compare with the original (c+di)? This transformation is called a
        _____ with factor ____ and center at _____.

    b. Distribute i(c+di). Write the result the usual way, with the real part first, and the imaginary part next. How does the resulting vector compare with the original? What is this transformation called?

    c. Distribute ki(c+di). Describe how the resulting vector compares with the original. This transformation does not have a name. It is the *composition* of the previous two.

In a previous course, you learned that $(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$.

4. Put that in words: to multiply complex numbers in polar form…

5. How would you use complex numbers to find the image of (a, b) by a *rotation* of θ° centered at the origin?

Complex Multiplication in Polar Form: A Specific Example

In Math 3, you learned that $(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$. We will use geometry to prove this, starting from the definition of i as the square root of –1.

0. Multiplying a complex number by i is the same as a _____ around _____

We will start by analyzing a specific example: (2+i) (3+4i)

1. a. Draw these two complex numbers as vectors on graph paper.
    b. Label the origin as O.
    c. Label the point 2+i as P, and 3+4i as Q.
    d. Label the point directly below P on the x-axis N.

2. Distributing, we see that $P \cdot Q = (2+i)Q = 2Q + iQ$
    a. Find the points for 2Q and iQ on your graph. Label them R and T.
    b. Find the point for 2Q+iQ, and label it U.

3. Explain: $P \cdot Q = U$

We'd like to show that the magnitude of U is the product of the magnitudes of P and Q, and that its angle is the sum of their angles. We will do this with the help of triangles ΔOUR and ΔOPN.

4. Using only the information about how we created U and R, show that ΔOUR and ΔOPN are similar. What is the scaling factor?

5. Show that $|U| = |P| \cdot |Q|$

6. Show that $\angle NOU = \angle NOP + \angle NOQ$

If you understand this so far, you're ready to generalize.

Complex Multiplication in Polar Form: Generalizing

This time we will go through the argument with a generic figure. P=a+bi, Q=c+di, and the other points are defined as in the figure below.

1. Explain this: (a+bi)(c+di) = a(c+di) + bi(c+di).

Another way to write the above is: (a+bi)Q = aQ + biQ if Q is the complex number (c+di).

By answering the next few questions, you will see that this figure is an illustration of the above equation.

2. On the figure, what point represents:
   a. aQ
   b. iQ
   c. biQ
   d. aQ + biQ
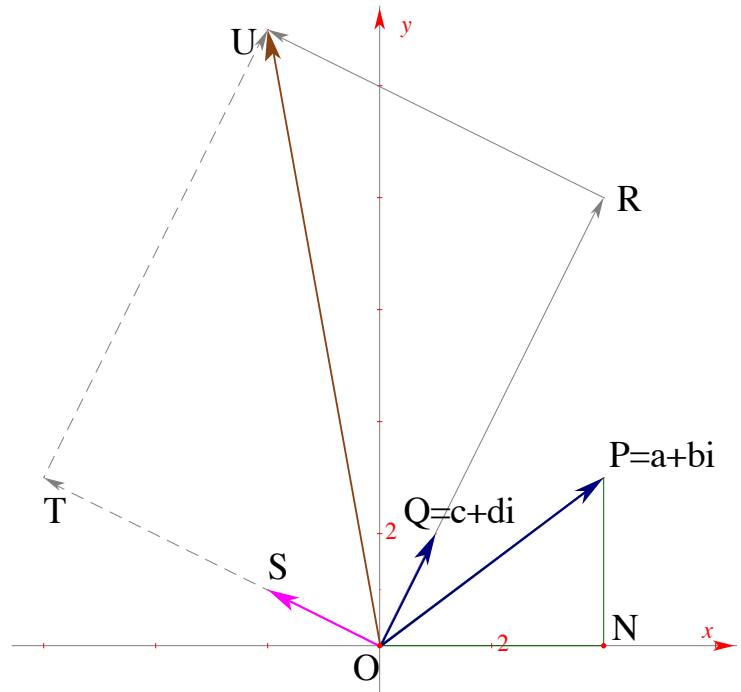
3. Explain why U = P·Q, using #2.

Up to this point, we worked with the rectangular form of P, Q, and U. To prove the result we are after, we will now switch to polar form, and use similar triangles in the figure above.

4. Say that P = (r_1, θ_1), Q = (r_2, θ_2), and U = (r, θ). Label the figure accordingly.

5. Prove that $\triangle ONP$ is similar to $\triangle ORU$. What is the scaling factor?

6. Prove that $r = r_1 r_2$.

7. Prove that $\theta = \theta_1 + \theta_2$.

Therefore: $(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$

QED

Computing Any Isometry Using Complex Numbers

1. **Review:** Make sure you know how to use complex numbers to compute the image of a point (a,b) under:
   a. Translation by a vector (v,w)
   b. Translation by a vector (r,θ)
   c. Rotation by θ degrees around the origin.

2. How would you use complex numbers to find the image of (a, b) by a rotation of θ° centered at (p, q)? (**Hint**: this will take three steps — translate, rotate, translate back)

What we are missing is a method for computing reflections.

3. The image of (a,b) after reflection in the x-axis is _____.

4. The image of a+bi after reflection in the x-axis is _____.

This image is called the conjugate of a+bi, which GeoGebra will return if you enter
   conjugate(a+bi).

Three-step method: We will find the image of (a,b) in a line y=mx, by rotating so the line lies on the x-axis, then we'll reflect across the x-axis, then we'll rotate back.

4. What is the angle between y = 2x and the x-axis?

5. What is the reflection of the point (3,2) across the line with equation y=2x?

6. Now let us reflect (3,2) in the line y=x.
   a. Predict the coordinates of the image
   b. Check whether the three-step method gives you that answer.

7. Work out a strategy (five steps) to reflect (3,2) across the line with equation y = 2x+5. Find the coordinates of the image.

8. Work out a strategy to glide-reflect (3,2) with mirror y=x+4 and vector (5,5)

You can now compute the results of any isometry! However this method has shortcomings:
◊ It is tricky to keep track of parentheses and order of operations.
◊ It is inconvenient when calculating the image of a polygon, as the calculation has to be done for each vertex.
◊ It does not generalize to three dimensions.
All of those concerns are addressed by using matrices. In fact, all computer animation is done by matrix calculations of geometric transformations. This is what we will learn next.

Computing Some Transformations with Matrices

We will represent points using a vertical (2 by 1) format: $\begin{bmatrix} x \\ y \end{bmatrix}$. An n-gon will be represented by

a 2 by n matrix. For example: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ represents a right triangle.

1. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a dilation

    centered at the origin, with factor k.

2. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in

    the y-axis.

3. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a reflection in

    the y=x line.

4. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a 180° rotation

    around the origin.

In a previous lesson, you found the formula for the image of a point (x,y) under a rotation of
θ° around the origin.

5. Find a matrix M such that $M \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the image of (x,y) after a rotation of θ°

    around the origin.

In a previous lesson, you found a way to use the complex conjugate as part of a three-step
process to reflect a point across any line through the origin.

6. Find a sequence of three matrices L, M, N such that $N \cdot M \cdot L \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ will correspond to the

    image of (x,y) after a reflection in the y=2x line. Note that the matrices are written from
    right to left in the multiplication.

Matrices in GeoGebra

GeoGebra can do matrix multiplication.

To enter a matrix, enclose it in {curly brackets}. Within that, each row is enclosed in curly brackets. Rows, and the items in each row, are separated by commas.

For example, to enter the multiplication $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, enter: {{1,0},{0,-1}}*{{2},{3}}

The result appears in the Algebra pane.

1. Check that the above multiplication gives the expected result.

If, for example, you want to use sin 56° in a matrix, just enter sin(56°). You don't have to get a decimal for it. However *you do have to use the degree symbol*, which you can get (in GeoGebra) as control-o.

However, it is tedious to enter matrices over and over, and it is easy make mistakes. One way to make this more convenient is to use the spreadsheet pane.

For example, to create this matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2. a. Enter these numbers into the spreadsheet:

	A	B
1	1	0
2	0	-1

   b. Select those four cells and choose the **Create Matrix** tool (in the pop-up menu below **Create List**).
   c. In the subsequent dialog, GeoGebra will ask you to name the matrix. Call it **rx**.
   d. Click on **Free Objects**, which means that the matrix will keep this value. (If you click on **Dependent Objects**, the matrix will change if you change the corresponding numbers in the spreadsheet.)

3. What does **rx** do?

In GeoGebra, you can transform any geometric object with this command in the Input Bar:
   ApplyMatrix[<Matrix>, <Object>]

4. Make a polygon, show the axes, and use **ApplyMatrix[rx, poly1]** to make sure everything is working. (Of course, if your polygon is not named **poly1**, use its actual name.)

Matrix-Making Machine

You can make a matrix that depends on a slider, and use it as a matrix-making machine. For example:

1. a. Make an Angle slider, called α, with
        Min -179°, **Max** 180°, **Increment** 1°, and **Width** 360 px.
   b. Create a matrix that depends on α, either in the Input Bar:
        ro = {{cos(α),-sin(α)},{sin(α),cos(α)}}
        or in the spreadsheet (you'll have to copy and paste the α character. Use **Dependent Objects**, since you want the matrix to change when you change the value of α in the slider.)
   c. You can then copy the matrix you made with this command:
        ro1=CopyFreeObject[ro]
(**ro1** is a possible choice for the matrix's name.) The value of the matrix will remain fixed, but **ro** will continue to depend on the slider. Keep it in order to manufacture as many rotation matrices as you need.

2. Use matrix multiplication to find the image of (2, 3) after a rotation of 22° around the origin.

3. The next three problems will use a 33° rotation. Create the matrix for it under the name **r**.

We don't yet know how to use matrix multiplication for translations, so you'll have to do translations yourself in the next two problems.

4. Use matrix multiplication to find the image of (2, 3) after a rotation of 33° around (5, 4)

5. Use matrix multiplication to find the image of (5, 4) after a rotation of 33° around (2, 3)

Matrices allow you to transform many points in a single calculation.

For example, to rotate (9, 4), (7, 5), and (8, 6) 33° around the origin, you can do $r \cdot \begin{bmatrix} 9 & 7 & 8 \\ 4 & 5 & 6 \end{bmatrix}$

6. What are the images of the three points?

More Practice

7. Make a matrix (perhaps named **ro45**) for a 45° rotation around the origin.

8. Make a matrix (perhaps named **ry**) for a reflection in the y-axis.

9. How would you use matrix multiplication to reflect a point (x_0, y_0) around a line through the origin, that makes an angle θ with the positive x-axis?
   a. Say this in words, not symbols
   b. Write the matrix multiplication, remembering to go from right to left.

10. Test your answer with the matrices you made in GeoGebra, using (2,3) for your pre-image, and 45° for θ since you know what the result of that reflection should be. *You will need another rotation matrix*, which you might call **ro45b.** Here is how you would enter this:

    D=ro45*rx*ro45b*(2,3)

    Make sure you got the answer you expected.

11. If a line has equation $y = 3x$, what angle does it make with the positive x-axis?

12. What is the image of (4, 5) across the line $y = 3x$?

Matrices and Translation

The shortcoming of using matrix multiplication in the way we have been is that it does not work for translations. This problem is solved if we represent points as 3 by 1 matrices, like this:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

1. Find a 3 by 3 matrix M such that $M \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the image of (x,y) after a

    translation by vector (v,w).

2. Find 3 by 3 matrices for
    a. Rotation around the origin by an angle θ
    b. Dilation centered at the origin, with factor k
    c. Reflection in the x-axis

3. Find a sequence of three matrices L, M, N such that $N \cdot M \cdot L \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the

    image of (x,y) after a θ° rotation around a point (p,q). Remember that the matrices are written from right to left in the multiplication.

4. Find a sequence of three matrices L, M, N such that $N \cdot M \cdot L \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to the

    image of (x,y) after a dilation centered at a point (p,q), with factor k.

5. Find a sequence of five matrices J, K, L, M, N such that $N \cdot M \cdot L \cdot K \cdot J \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ will correspond to

    the image of (x,y) after a reflection across the line y=mx+b.

We are now able to calculate the matrices for any dilation or isometry. (For a glide reflection, we can just multiply the matrices for the corresponding reflection and translation.) With the help of technology, those calculations are instantaneous, so we have basically solved the problem of computing images.

Building a GeoGebra Matrix Factory

In order to get more practice with transformation matrices, we will create a GeoGebra matrix factory for 3 by 3 matrices. The basic matrices we need are:

  Translation by a given vector, often ending at the origin, and its inverse
  Rotation around the origin, by a given angle, and its inverse
  Dilation centered at the origin
 Reflection across the x-axis

As you learned in previous lessons, by combining these, you can compute the output of all the main transformations. Once these are set up, all we need to do to create a single matrix to do a given job is multiply the appropriate basic matrices in the right order.

Setup

Download the file **3-by-3.ggb**. If you want to create it yourself, here are the ingredients:
  - The axes and the grid are shown. The latter is spaced with x and y distances of 1.
  - A vector **v** for the translations, preferably starting at a lattice point, and ending at the origin, but both endpoints are free to move, as is the vector itself.
  - An angle slider α, for the rotations, from -179° to 180°, with length 360 px.
  - A slider **k** for the dilations, with the default parameters.
  - An asymmetric polygon to be used as pre-image when testing the matrices you create, for example a scalene right triangle called **tri**.

There is also a line **l**, and the angle β it makes with the x-axis, but those will not be needed right away.
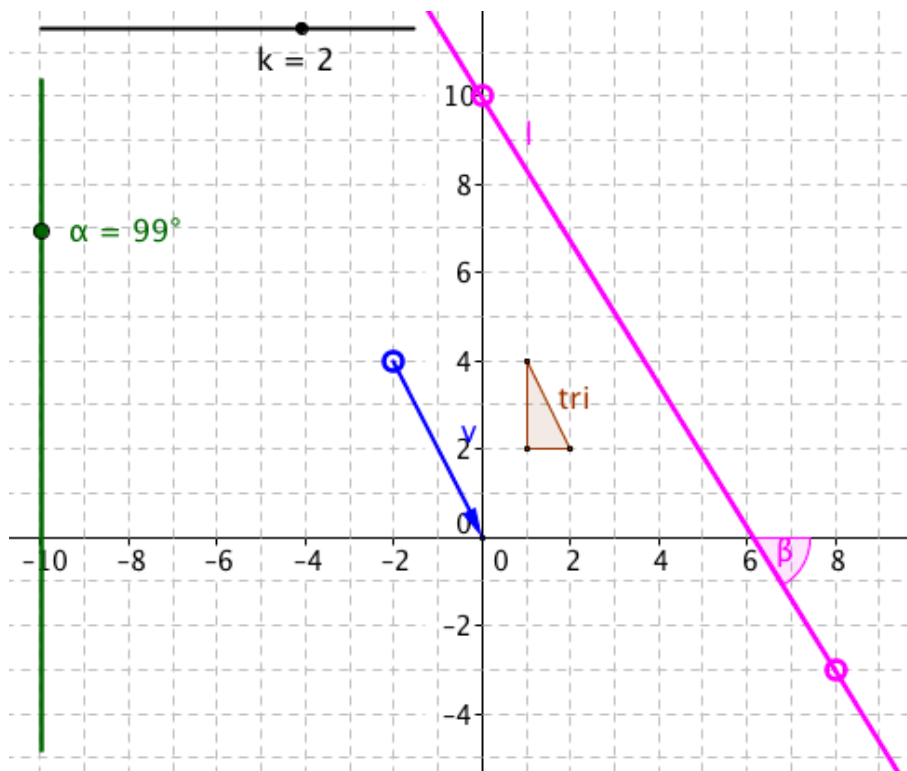
Making the Basic Matrices

Suggestions:
- To save time and avoid mistakes when typing brackets, you can copy and paste an existing matrix definition, and edit it. Or you can use the spreadsheet and the **Create Matrix** command.
- Test each matrix as soon as you make it, by using **ApplyMatrix[<matrix>,tri]**

x(v) and **y(v)** will give you the x- and y-components of vector **v.**

1. Make a translation matrix for **v**, and its inverse. Call them **tra** and **trb**.

2. Make a rotation matrix, center at the origin, angle α, and its inverse. Call them **roa** and **rob**.

3. Make a dilation matrix, centered at the origin. Call it **dil**.

4. Make a matrix for reflection in the x-axis. Call it **rex**.

You now have all you need to quickly create matrices for any of the main transformations.

Matrix Combinations

Example: Say you want to create a single matrix to rotate objects by 56° around (3,4). We'll want to translate the center to the origin, rotate, then translate back.
   a. Place the tail of **v** at (3,4), and its head at the origin. This changes **tra** and **trb** to the matrices we will need.
   b. Set α to 56°. This changes **roa** to the needed rotation matrix.
   c. Enter **mat=trb*roa*tra** on the input line. **mat** will be the needed matrix!
   d. Place **tri** at a convenient location.
   e. Enter **ApplyMatrix[mat,tri]** on the input line to test your result.
If everything is at it should be, enter **rot1=CopyFreeObject[mat]** on the input line to preserve a copy of the matrix. Then you can reuse **mat** for future compositions.

5. Make a matrix for a 120° rotation around (4,3).

6. Make a matrix for a dilation centered at (1, -3) with factor 2.

Making a matrix to reflect in a line is a bigger project, as you need five matrices. Moreover, it requires using the angle between the line and the x-axis, and therefore our slider-based rotation matrices will not be as convenient. Still, it's not too difficult if you use line **l** and create rotation matrices for angle β.

7. Make a reflection matrix for line **l**.

This completes your matrix-making factory. (However it is certainly possible to improve it!)

Review: From Complex Numbers to Matrices

A point in the plane can be thought of as a _____ number, just like a point on a number line can be thought of as a _____ number.

1. Write the following three famous points in a+bi (rectangular) form:
   a. (1, 45°)
   b. (1, 60°)
   c. (1, 90°)
   For the purpose of the following exercises, call these points respectively s, t, and i.

2. s, t, and i are all on a certain geometric figure. What am I referring to?

3. Using your calculator, or not, compute the following, and show the results on graph paper:

   a. s+t   b. t+i   c. s+i

   d. s·t   e. t·i   f. s·i

4. Describe the results of the computations above, using the words *translation* or *rotation*. (There are two ways of doing this for each example.)

5. Explain how to use complex numbers to translate a point (x, y) by a vector (v, w)

6. Write the coordinates of the point (1,θ) in a+bi form.

7. Explain how to use complex numbers to rotate a point (x, y) by an angle θ around the origin.

8. Use the answer to the previous problem to explain where the rotation matrix comes from.