Activities

Lew Douglas

Pentagrams and Spirals

Kepler called the golden ratio one of geometry's two "great treasures"

TEACHER'S GUIDE

The inspiration for this activity is the mathematics videotape *Donald in Mathmagic Land,* originally produced as a movie by the Walt Disney Company in 1959. Anticipating current music videos and some commercials, the images are tantalizing and rapid, encouraging some teachers to show it to students as early as third or fourth grade. The mathematical presentation, however, is deep and brilliant, so I had a natural desire to slow it down and make it truly accessible to students. I generally show the videotape to students just after we complete the exercises presented in this article, and several have told me that it was the third or fourth time they had seen it but the first time they understood it.

The golden ratio, a topic prominently featured in the videotape, first appears in Euclid's *Elements* as Proposition 30 in Book VI: "to cut a given finite straight line in extreme and mean ratio" (Heath 1959, 267). It may have even been known before the time of the Greeks. Leonardo da Vinci referred to the *sectio aurea,* "golden section," and Kepler referred to it as the *sectio divina,* "divine section." "Geometry has two great treasures," said Kepler. "One is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio" (Huntley 1970). Mathematics teachers may be more familiar with the golden ratio in its more modern incarnations, perhaps from reading *Fascinating Fibonaccis* (Garland 1987). It appears as an "excursion" or "extra" in many geometry textbooks, its appeal being the relatively easy connection to architecture and natural phenomena. Ironically, however, the classic textbook introduction is often algebraic and unnatural. I regretfully must add that the power of the pentagram as a symbol is still with us: a friend told me that she could not use this activity in her school because the pentagram was used as a symbol by a local gang.

Grade levels: 9–11

Objectives: To introduce students to a fascinating application of basic geometric and algebraic ideas

Prerequisites: Success on these worksheets depends on students' familiarity with regular polygons, similar triangles, finding missing angles in triangles and intersecting lines, and the quadratic formula. Teachers who use these worksheets with classes that have not covered these topics will need to modify the exercises. At our school, students solve the key quadratic equation (sheet 2, question 6) several times for homework *before* tackling these sheets because the "phantom" 1's for the coefficients, $a = 1$, $b = -1$, and $c = -1$, can be confusing.

Directions: The sheets should be done in order, although not necessarily on consecutive days. They were designed for students working in cooperative groups, and the entire set should take one or two forty-five- to fifty-five-minute class periods. Some terms are left for the students to fill in to make the reading more active; I tried to choose them so that the task would not take too much time. Sheets 1 and 2 are prerequisites to sheet 3, so they should be completed and checked before sheets 3 and 4 are distributed. Moreover, having sheet 3 prematurely

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This section is designed to provide in reproducible formats mathematics activities appropriate for students in grades 7–12. This material may be reproduced by classroom teachers for use in their own classes. Readers who have developed successful classroom activities are encouraged to submit manuscripts, in a format similar to the "Activities" already published, to the senior journal editor for review. Of particular interest are activities focusing on the Council's curriculum standards, its expanded concept of basic skills, problem solving and applications, and the uses of calculators and computers.

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would allow students to look up some answers to sheet 2.

Sheet 1: This sheet familiarizes students with the pentagram and contains some interesting systematic counting exercises, but the only results needed later are the answer to question 1 and an awareness of the similar triangles that can be found in the figure. Some students will assume—correctly, as it turns out—that $m\angle APE$ is trisected by $\overline{PT}$ and $\overline{PN}$. Although proving this assumption makes a good exercise, an extended discussion is not appropriate at this time. Instead, consider doing a formal proof beforehand, coming back to it later, or ignoring the issue entirely unless the students bring it up.

If symmetry has previously been discussed, using it will make the counting for questions 5–8 much easier and will supply a check on the answers, all of which are multiples of 5. If not, here is not the place to begin because of time constraints. However, systematic counting should be encouraged.

Sheet 2: Questions 1–8 lead students to the computation of the golden ratio both in radical form (question 7) and as a decimal approximation (question 8). Students might need help to identify which of the two solutions in radical form is the positive one. Nothing more powerful than a scientific calculator with a single memory location is required. Ideally, each student will have a calculator, both to practice correct use and to minimize errors, but one calculator for each group is workable. Question 9 of sheet 2 is pivotal; the teacher must either check each group or go over the answers in a whole-class discussion before proceeding. A suggestion is for each group to write sequentially one result on the chalkboard for the whole class, continuing in this manner until all the relationships have been found. The use of roman numerals makes it easier to keep track of which segments are being discussed and their relative sizes. The addition and division relationships derived here are the heart of the matter and are illustrated beautifully in *Donald in Mathmagic Land* by a star that comes apart as it rolls.

Sheet 3: Question 1 begins with a review of questions 7 and 8 on sheet 2. Students delight in noticing that the decimal part of the golden ratio does not change when they press the "square" or "reciprocal" key on their calculators, so observe carefully as students work on the end of sheet 2. Have a student who has discovered one of these facts share this insight with the class, or give hints as needed. A scientific calculator for the overhead projector is useful here but is certainly not required. Here is a good way to summarize: With φ in the display, press x^2, $\sqrt{x}$, 1/x, 1/x in that order. Parts 2–4 of question 1 show that these relationships, surprising as they are, are mandated by the defining equation. More observant students may notice that the equation they solve in question 4 is exactly the same one that they solved in question 7 of sheet 2. If they do, be sure to have them share this insight with the class; if not, ask, "How does the work you are doing relate to the work on the previous sheet?"

The rest of sheet 3 relates these algebraic relationships to the defining *geometric* property of a golden rectangle: removing or adding an appropriate square gives yet another golden rectangle. This property is illustrated nicely in *Donald in Mathmagic Land* as the shorter side of the rectangle rotates twice, around different points, to form the square.

Sheets 4A and 4B: These sheets lead students through the straightedge-and-compass construction of a golden spiral. If students do not know what a spiral is, discuss this concept at the beginning. Using the diagonals as an accuracy check is crucial; in fact, using the diagonals *instead* of a compass is quite satisfactory. If only a compass is used, however, the construction errors become significant after only two or three squares are removed. As before, the blanks in questions 2–4 serve to make the reading more active.

Extensions: Many follow-up assignments and problems are available, for example, this particularly interesting connection with Platonic solids:

Take three mutually perpendicular, congruent golden rectangles with a common center. Their twelve vertices are the vertices of a regular dodecahedron! A simple model can be made from three large index cards, which are approximate golden rectangles. Make a slit equal in length to the short side of two cards in the center of the cards parallel to their long sides. Insert one card into another. On the third card, extend the slit to one edge and slip it over the long side of the exterior card and through the center of the interior card. Punch holes near each vertex and weave brightly colored yarn through the holes to make the edges of the dodecahedron. See the photograph.

Connections with Fibonacci numbers, art, architecture, music, natural phenomena, regular decagons, and Platonic solids can be explored, possibly as part of a mandatory or optional research project. Students find it interesting to see a seemingly unrelated problem, such as the consideration of ratios of consecutive terms of Fibonacci-like sequences starting with *any* two terms, lead unexpectedly to the now familiar golden ratio. A rich history of appreciation is evident, with mathematicians from Plato to Kepler to da Vinci waxing eloquent about this mathematical phenomenon. Students can be asked to research the origin of *golden* or *divine,* terms they may find surprising in this context. For further project ideas, see the resources in the **bibliography.**

Assessment: This activity itself can serve as an assessment of students' understanding of such concepts as regular polygons, facts about angles, and compass-and-straightedge constructions, as well as of such techniques as recognizing and using similar triangles and solving quadratic equations.

To assess students' grasp of concepts related to the golden ratio, request that they include the main mathematical ideas from this activity as part of a report, such as those suggested previously.

Answers: Sheet 1: regular, diagonals. 1. The smaller acute angles are 36 degrees, the larger acute angles are 72 degrees, and the obtuse angle is 108 degrees. Students may use what they know about regular polygons, vertical angles, isosceles triangles, or even inscribed angles. 2. 36 degrees, 72 degrees, 108 degrees. 3. None. 4. 36°-72°-72°: $\triangle PGR$, $\triangle PBE$, $\triangle PTN$; 36°-36°-108°: $\triangle MTN$, $\triangle PAE$. Other answers may be congruent copies of these answers. 5. Two. 6. Five. 7. $\triangle PGR$: five copies, $\triangle PBE$: ten copies, $\triangle PTN$: five copies, $\triangle MTN$: five copies, $\triangle PAE$: ten copies. 8. Thirty-five.

Sheet 2: 1. Four. See question 9. 2. $PG = AG = 1$, $GR = x - 1$. Students having trouble with GR need to recall that $\triangle APR$ is isosceles. 3. $\triangle PGR$ or $\triangle PRG$; because corresponding angles are equal. 4. Answers will vary. 5. $1/(x - 1)$. 6. $x^2 - x - 1 = 0$. 7. $x = (1 + \sqrt{5})/2$. 8. $x = 1.618033989\ldots$; some calculators may differ slightly. 9. I. 0.618033989 . . . . II. 1. III. 1.618033989 . . . . IV. 2.618033989 . . . . 10. Six ratios should be found. The most important observations follow: IV/III = III/II = II/I = 1.618033989 . . . ; I + II = III; and II + III = IV. 11. Answers will vary.

Sheet 3: 1. $\varphi^2 = 2.618033989\ldots$; $1/\varphi = 0.618033989\ldots$. 2. $\varphi^2 = \varphi + 1$; $1/\varphi = \varphi - 1$. 3. Multiply both sides of the second equation by φ. 4. $\varphi = (1 + \sqrt{5})/2$. 5. $EO = 1$; $ED = \varphi - 1$. 6. The rectangles are similar. To prove it, show that corresponding angles are equal and that corresponding sides are proportional. Those students used to dealing only with similar *triangles* may need some help. 7. Another golden rectangle.

Sheet 4: 1. Another golden rectangle. 2. Golden rectangle; golden rectangle. 3. Similar; proportional; ED. 4. GD; PD; collinear.

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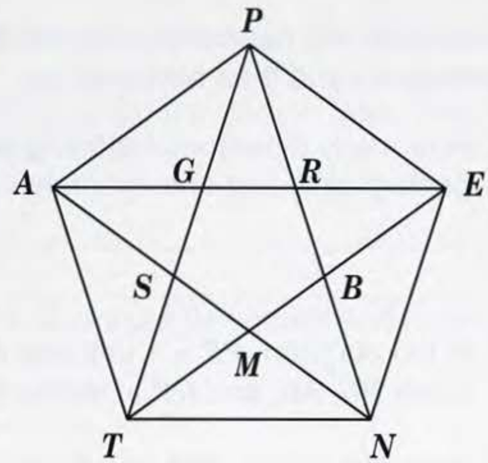
(Worksheets begin on page 683)



The figure to the right is called a pentagram. It was made by drawing the large __________ pentagon ***PENTA*** and all its __________. The pentagram was chosen by the Pythagoreans as the symbol of their brotherhood because of its many interesting properties. It has also been used by many cultures as a symbol of magic and the occult. This sheet will familiarize you with the triangles in the pentagram.

1. Without measuring, determine the measures of the angles in $\triangle AGP$ and $\triangle PGR$. Check your answers with those of your neighbor. Write your reasons here.

   ____

   ____

2. Determine the measures of a few more angles in the figure. What angle measures occur in the pentagram?

   ____

3. Of all triangles in the pentagram, how many are *not* isosceles? __________

4. The shape of a triangle is determined by its angles. For example, some of the triangles in the figure could be called 36°-72°-72° triangles. What other angle combinations are possible? Name as many different-sized triangles as you can for each shape and compare your results with those of your neighbor.

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   36°-72°-72° : One answer is $\triangle PGR$; others are

   ______________________________.

5. How many different-shaped, or dissimilar, triangles can you find in the pentagram? __________

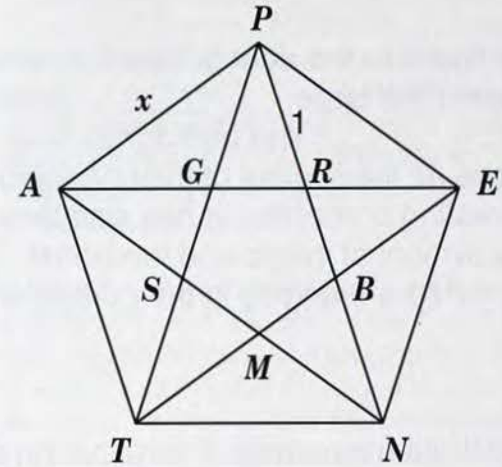
6. Look again at your answer to question 4. How many different, that is, noncongruent, triangles can you find in the pentagram? __________

7. In question 4, you listed one example of each different triangle. How many copies of each can you find in the pentagram? __________

   $\triangle PGR$: One solution is five copies; others are __________.

8. How many different triangles can you find altogether? Find a systematic way to count them. Check with your neighbor to be sure. __________

From the *Mathematics Teacher*, November 1996



This sheet will familiarize you with the segments in the pentagram and their relationships.

1. How many different-length segments are in the pentagram? List one example of each length.

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2. In the diagram, $PR = 1$ unit and $AP = x$. Label PG, AG, and GR in terms of x and 1.

3. Consider only $\triangle APR$ and the two triangles into which it is split by PG. $\triangle APR$ is similar to which other triangles? _______________

   Why? ____

4. Sketch these two similar triangles so that they are both in the same position. Label all side lengths and angles. Check with your neighbor to be sure that you are right.

5. Complete this proportion:

   $\frac{x}{1} = \_\_\_\_\_\_\_.$

6. Put the equation from question 5 in standard quadratic form, that is, with zero on one side of the equation. _______________________________

7. Use the quadratic formula to solve for x. Do not use a calculator yet; leave your answer in radical form. Remember that x is positive because $x = AP$. _______________

8. Use your calculator to find a decimal approximation for x. Copy all decimal places shown on your calculator: _______________. Store this value in your calculator's memory.

9. In question 1, you listed one example of each different-length segment in the pentagram. Express each length in terms of x, then compute it using your stored value. Copy *all* decimal places shown on your calculator.

   I. GR = __________ II. AG = __________ III. AR = __________ IV. AE = __________

10. Compute *all* the ratios of larger lengths to smaller lengths in the list above. Write any interesting relationships that you notice in terms of the roman numerals.

    ____

11. Write any simple addition relationships that you notice in terms of the roman numerals.

    ____

These interlocking addition and division relationships are what make the pentagram so special.

On this sheet you will explore ratios in the pentagram.

1. The most common ratio that you found in question 9 of sheet 2 is called the *golden ratio*. We use the Greek letter phi, ϕ, to name it. On sheet 2, we called it $\boldsymbol{x}$. Thus

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618033989\ldots.$$

You already stored φ in your calculator in question 8 of sheet 2. Compute its square and reciprocal and record the results below.

$$\varphi^2 = \underline{\hspace{4cm}}. \qquad \frac{1}{\varphi} = \underline{\hspace{4cm}}.$$

2. Record what you notice by expressing φ^2 and $1/\varphi$ in terms of φ and 1.

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3. Show that these equations are equivalent. ______________________

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4. Solve the easiest equation in question 2 for φ exactly. ______________________

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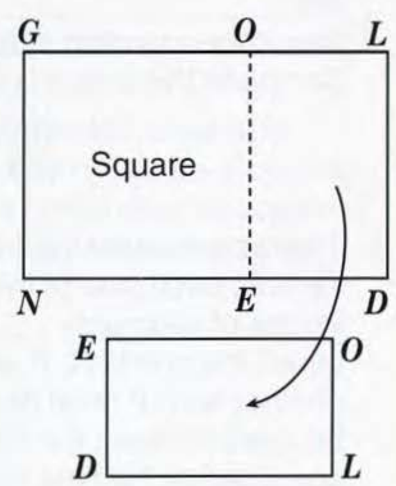

5. Recall from question 9 of sheet 2 that φ is the ratio of the length of any segment in the pentagram to the length of the smaller segment. If the ratio of length to width in a rectangle is φ, it is called a "golden" rectangle. The diagram shows a golden rectangle with a square removed. The remaining rectangle has been turned 90 degrees and redrawn below the original. Since we are only concerned with ratios, we can let $GL = \varphi$ and $GN = 1$. Label all lengths in both diagrams accordingly.

6. Make and prove an interesting conjecture about rectangles ***GLDN*** and ***EOLD***. ______________________

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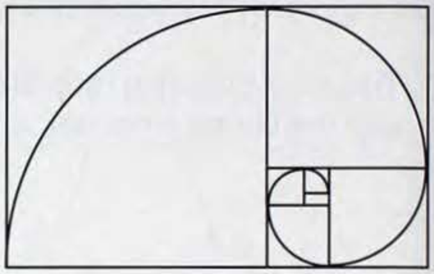
7. This concept is quite remarkable. If you start with a golden rectangle and remove a square built on the shorter side, what remains is another golden rectangle! What do you get if you start with a golden rectangle and add a square built on the larger side?

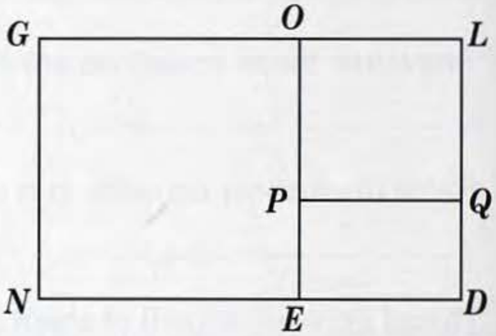
The golden rectangle's ability to reproduce itself leads to another remarkable shape, as you will see on sheets 4A and 4B.

On this sheet you will learn how to construct a golden spiral, like the one in the figure, using only a straightedge and compass.



1. As you saw in question 7 of sheet 3, if you start with a golden rectangle and remove a square built on the shorter side, what remains is another golden rectangle. What do you get if you remove a square built on the shorter side of this new, smaller golden rectangle? ______________________

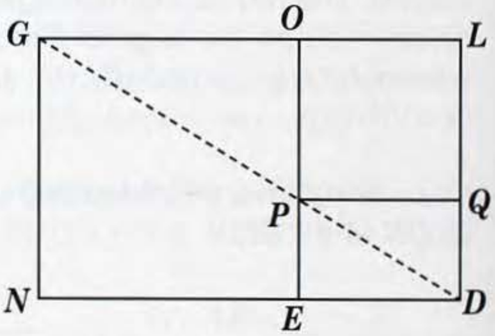
   In theory you can then use a compass to construct an infinite spiral of smaller and smaller golden rectangles by performing this process over and over. But the construction errors mount up and quickly ruin the picture. Why? We need an accuracy check, and fortunately, a simple one is available.



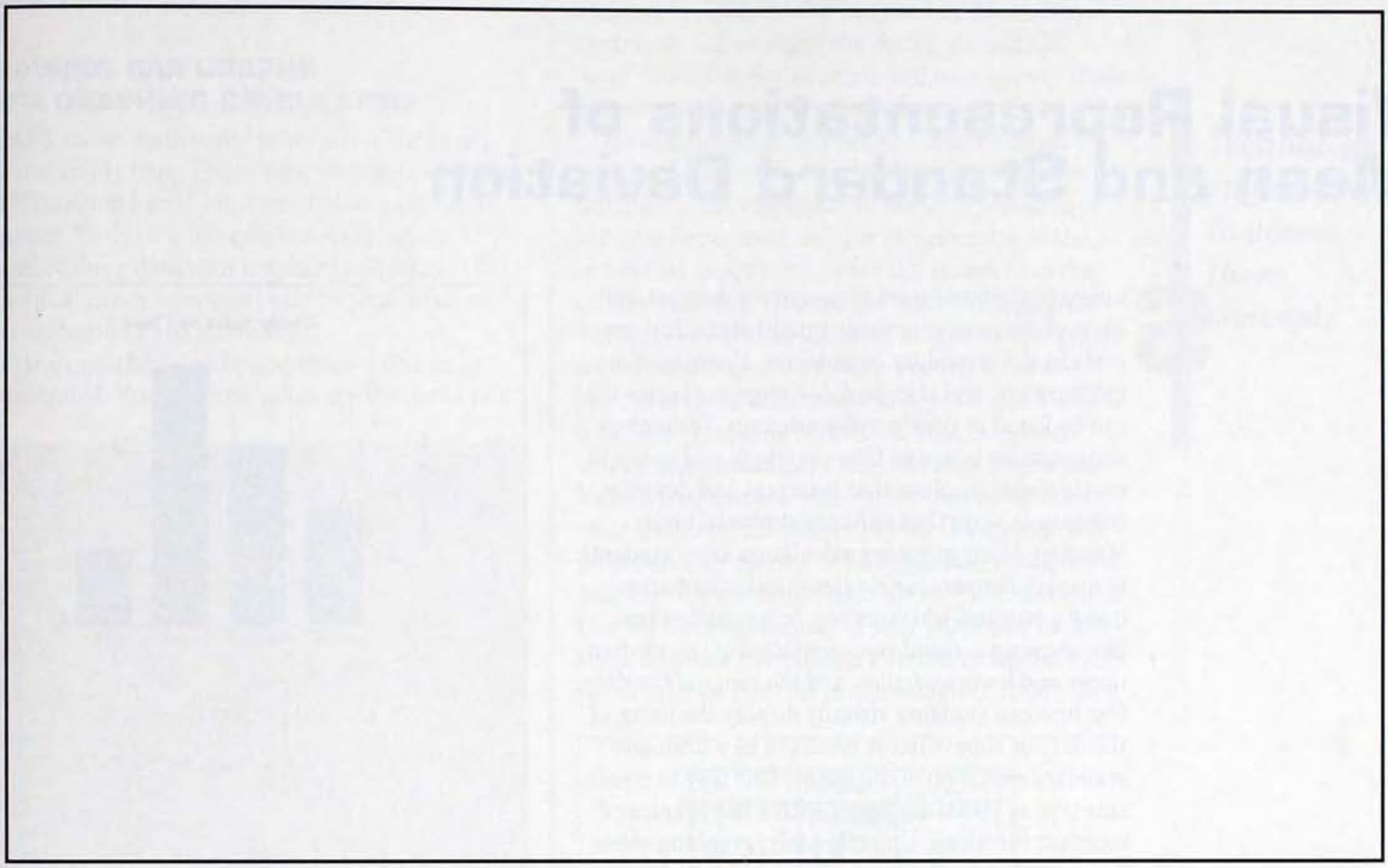
2. Start with golden rectangle ***GLDN*** and remove square ***GOEN.*** Then ***OLDE*** is a ______________________.
   Next remove square ***OLQP.*** Then ***PQDE*** is a
   ______________________.

3. Since ***GLDN*** and ***PQDE*** are both golden rectangles, they are ______________________. Therefore, their corresponding sides are ______________________.
   Complete this proportion:

$$\frac{GN}{ND} = \frac{PE}{\_\_\_}.$$



4. If we assume that the sides of ***GLDN*** are horizontal and vertical, each side of this proportion is a slope. Since the slopes of segments __________ and __________ are equal, the points ***G***, ***P***, and ***D*** must be ______________. Put another way, ***P*** must lie on diagonal ***GD***. This property can be used to check the construction of point ***P***, which will be described in the next question.



5. Start with this golden rectangle. Use your compass to locate the points needed to remove squares in a spiral fashion: from left, then top, then right, then bottom. Lightly drawn diagonals ***GD*** and ***EL*** will locate the subsequent horizontal and vertical lines as a check on your compass work. Add quarter circles centered at ***E, P,*** and so on, to duplicate the diagram at the top of sheet 4A.