

Complex Numbers

This unit marks the end of a journey that students started in Kindergarten: the discovery of ever more inclusive sets of numbers. The following lessons from *Algebra: Themes, Tools, Concepts* can set the stage:

11.2 Decimals and Fractions,
11.4 Irrational Numbers, and
14.7 Equations and Numbers.

In fact, the first page of the unit is a slightly edited version of the end of *ATTC* 14.7.

The approach is unusual, in that it starts not from the solution of quadratic equations, but from a graphical, geometric approach. (This is inspired by *Algebra II/Trigonometry: A Guided Inquiry*, by Stein, Crabill, and Chakerian, an out-of-print textbook with many fine ideas.)

The worksheets are dense with ideas, and definitely need to be accompanied by substantial class discussion, teacher explanations, and additional practice and review. Nevertheless, they represent a coherent approach that makes pedagogical sense and helps review important skills.

What the unit does not include, (because in my view it would be a bit much for Algebra 2,) is a proof that the geometric definition given for complex number multiplication in polar form is consistent with the distributive rule using rectangular form. If you are interested in the proof, find it in the **Computing Transformations** document on this Web site.

—Henri Picciotto

Kinds of Numbers

The first numbers people used were whole numbers. It took many centuries to discover more and more types of numbers. The discovery of new kinds of numbers is related to the attempt to solve more and more equations. The following equations are examples.

a. $x + 2 = 9$	b. $x + 9 = 2$
c. $2x = 6$	d. $6x = 2$
e. $x^2 = 9$	f. $x^2 = 10$
g. $x^2 = -9$	

1. Pretend you only know about the *natural numbers*. (These are the positive whole numbers.) List the equations above that can be solved.
2. Pretend you only know about the *integers*. (These are positive and negative whole numbers, and zero.) List the equations above that can be solved. Find one that has two solutions.
3. Pretend you only know about the *rational numbers*. (These are all fractions, positive, negative, and zero. Of course, integers are included, since for example $3 = 6/2$.) List the equations above that can be solved.
4. The *real numbers* include all rational and irrational numbers. Which equations can be solved if you know about all the real numbers?

Natural numbers, integers, rational numbers, and real numbers can all be found on a one-dimensional number line. However, to solve equation (g), you need to get off the number line! The solution is a *complex number*, and it is written $3i$. The number i is a number one unit away from 0, but off the number line. It is defined as a number whose square is -1 :

$$i^2 = -1.$$

Complex numbers cannot be shown on a line. They require a two-dimensional number plane. You will learn more about them in this class.

5. Create an equation whose solution is:
 - a. a natural number
 - b. an integer, but not a natural number
 - c. a rational number, but not an integer
 - d. an irrational number
6. Create an equation that has no real number solution.

Complex Numbers

The real numbers can be represented by the *real number line*. That line is the x-axis of the *complex plane*. The points of the complex plane can be thought of as *complex numbers*.

Complex numbers can be identified by their rectangular coordinates, or by their polar coordinates. For example, $(1, 1)$, or $(\sqrt{2}, 45^\circ)$ is a complex number. A generic complex number z could be written as (a, b) , or (r, θ) . Even real numbers can be thought of as complex numbers. For example, -1 is $(-1, 0)$, or $(1, 180^\circ)$. However, for complex numbers to really be numbers, we need to define operations.

Addition and Subtraction

Your teacher will explain how to add complex numbers *as vectors*.

- Use graph paper sketches to add the following complex numbers, shown in rectangular form:
 - $(1, 2) + (4, 3)$
 - $(-1, -2) + (4, -3)$
 - $(6, 2) + (3, 1)$
- Use graph paper sketches to add the following complex numbers, shown in polar form. Give your answer in either polar or rectangular form.
 - $(2, 0^\circ) + (5, 180^\circ)$
 - $(3, 180^\circ) + (3, 270^\circ)$
 - $(1, 60^\circ) + (7, 60^\circ)$
- Does the vector addition of complex numbers work as we expect it to when we use it on real numbers (thinking of them as complex numbers)? Explain, with a few examples.
- For any x on the real number line, $x + 0 = x$ and there exists an opposite $-x$ such that $x + (-x) = 0$.
 - Do complex numbers have a zero? If yes, what is it? (Give both rectangular and polar form.)
 - Does each complex number have an opposite? How do you find it? (Explain in both rectangular and polar form.)

To subtract a complex number from another, you add its opposite.

- Use graph paper sketches to subtract the following complex numbers, shown in rectangular form:
 - $(1, 2) - (4, 3)$
 - $(-1, -2) - (4, -3)$
 - $(6, 2) - (3, 1)$
 - $(-6, 2) - (6, -2)$
 - $(0, 5) - (5, 0)$
 - $(0, 7) - (0, -3)$
- Use graph paper sketches to subtract the following complex numbers, shown in polar form. Give your answer in either polar or rectangular form.
 - $(2, 0^\circ) - (5, 180^\circ)$
 - $(3, 180^\circ) - (3, 270^\circ)$
 - $(1, 60^\circ) - (7, 60^\circ)$
 - $(1, 60^\circ) - (1, 300^\circ)$
 - $(1, 60^\circ) - (1, 240^\circ)$
 - $(4, 10^\circ) - (6, 110^\circ)$

The mirror image of a complex number, across the real number line, is called its *conjugate*.

- Add each number to its conjugate:
 - $(0, 3)$
 - $(2, 0)$
 - $(2, 3)$
 - $(4, 0^\circ)$
 - $(5, 270^\circ)$
 - $(6, 60^\circ)$
- Add (a, b) to its conjugate. Repeat with (r, θ)

Multiplication of Complex Numbers

To multiply two complex numbers in polar form, you multiply their radii, and you add their angles.

$$(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 \cdot r_2, \theta_1 + \theta_2)$$

This is a surprising idea, but you will soon see that it really works.

- Multiply these complex numbers given in polar form:
 - $(2, 0^\circ) \cdot (5, 180^\circ)$
 - $(3, 180^\circ) \cdot (3, 270^\circ)$
 - $(1, 60^\circ) \cdot (7, 60^\circ)$
- Multiply these complex numbers given in rectangular form. Give your answer in either polar or rectangular form.
 - $(1, 1) \cdot (2, 2)$
 - $(0, 3) \cdot (-4, 0)$
 - $(\sqrt{3}, 1) \cdot (1, \sqrt{3})$
- Does multiplication of complex numbers work as we expect it to when we use it on real numbers (thinking of them as complex numbers)? Explain, with a few examples.
- For any x on the real number line, $x \cdot 1 = x$. If $x \neq 0$, there exists a reciprocal $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$.
 - Do complex numbers have a one? If yes, what is it? (Give both rectangular and polar form.)
 - Does each complex number except 0 have a reciprocal? How do you find it? (Use polar form.)
- Find the square roots of the following complex numbers. Hint: each one has two square roots.
 - $(1, 180^\circ)$
 - $(2, 180^\circ)$
 - $(4, 180^\circ)$

The complex number $(1, 90^\circ)$ is called i .

- i is one of the square roots of _____. In rectangular form, $i =$ _____.

Notation in Terms of i

Complex numbers in rectangular form are usually written in terms of i . For example:

$$(0, 2) \text{ is written } 2i$$

$$(0, -1) \text{ is written } -i$$

$$(2, 3) \text{ is written } 2+3i$$

Addition of complex numbers is easy in this notation, because it amounts to adding like terms. Example:

$$(2 + 3i) + (-3 + 2i) = -1 + 5i$$

- To check that this method of multiplication also works, redo the problems in #2 this way.
Example: #2a.

$$(1 + i)(2 + 2i) = 2 + 4i + 2i^2$$

$$\text{but } i^2 = -1, \text{ so}$$

$$(1 + i)(2 + 2i) = 2 + 4i - 2 = 4i$$

In polar form, $4i = (4, 90^\circ)$, which is the answer you should have for this problem.

Powers and Roots of Complex Numbers

Squares

- Square these numbers:
 - $(1, 180^\circ)$
 - $(2, 180^\circ)$
 - $(4, 180^\circ)$
 - $(2, 0^\circ)$
 - $(3, 270^\circ)$
 - $(1, 60^\circ)$
- Square these numbers:
 - $1 + i$
 - $-1 - i$
 - $2 - 3i$
- Find another number which has the same square as $2 - 3i$.

n^{th} Powers

- Calculate i^2, i^3, i^4 , etc. Describe the pattern.
- Let $z = (1, 30^\circ)$. Calculate z^2, z^3, z^4 , etc. Describe the pattern.
- Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. Calculate z^2, z^3, z^4 , etc. Describe the pattern. (Hint: it's a lot easier in polar form!)
- Let $z = 1 + i$. Calculate z^2, z^3, z^4 , etc. Describe the pattern.

n^{th} Roots

- Find the cube roots of 1. Hint: there are three of them.
- Find the fourth roots of 1. Hint: there are four of them.
- Find the fifth roots of 1.

Every complex number has n n^{th} roots.

- Find the fifth roots of $1 + i$.

Complex Number Review

Student-created problems

- Next to each complex number, write A if the point is on one of the axes, and S if it's related to a special right triangle. If it is not on one of the axes, indicate the quadrant it's in by writing I, II, III or IV.
 - 5
 - $3 - 4i$
 - $2 + 7i$
 - $5 - 2i$
 - $8 - 3i$
 - $7 + 8i$
- Convert them to polar form.
- Next to each complex number write A if the point is on one of the axes, and S if it's related to a special right triangle. If it is not on one of the axes, indicate the quadrant it's in by writing I, II, III or IV.
 - $(7, 100^\circ)$
 - $(15, 36^\circ)$
 - (r, q)
 - $(3, 270^\circ)$
 - $(5, 135^\circ)$
 - $(7, 10^\circ)$
- Convert them to $a + bi$ form.

In problems #5-7, give your answers in either polar or $a+bi$ form.

- Add.
 - $(7 + 10i) + (2 + 3i)$
 - $(3, 180^\circ) + (5, 270^\circ)$
 - $(6 + 8i) + (5 - 2i)$
 - $(3, 195^\circ) + (5, 35^\circ)$
 - $(1, 225^\circ) + (4, 20^\circ)$
 - $(5 + 2i) + (3 + 19i)$
- Multiply.
 - $(3, 50^\circ) \cdot (7, 360^\circ)$
 - $(4 + 7i)(2 + 2i)$
 - $(5 + 2i)(3 + 19i)$
 - $(4, 225^\circ) \cdot (-5, 3^\circ)$
 - $(5 + 3i)(10 - 2i)$
 - $(2, 150^\circ) \cdot (0, 210^\circ)$
- Find both square roots.
 - $7 + 2i$
 - $4 + 10i$
 - $(7, 60^\circ)$